Homework 1

Due date: Feb 16, 2021, 11:55PM.

Bibliography: Trefethen & Bau, Lectures 1-8. Problems 1-4 = 1 pt each, Problem 5 = 4 points.

1. Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ with SVD $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$. Compute the SVDs of the following matrices in terms of $\boldsymbol{U}, \boldsymbol{\Sigma}, \boldsymbol{V}$:
   a) $(\boldsymbol{A}^T\boldsymbol{A})^{-1}$
   b) $(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$
   c) $\boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A})^{-1}$
   d) $\boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$
2. Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ with SVD $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$. Define the Moore-Penrose pseduo-inverse $\boldsymbol{A}^+ = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$. Show that
$$\boldsymbol{A}^+ = \arg \inf_{\boldsymbol{X} \in \mathbb{R}^{m \times n}} \|\boldsymbol{A}\boldsymbol{X} - \boldsymbol{I}\|_F.$$
   What is the value of the infinum?
3. Prove:
   a) $\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A}$
   b) $\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+$
   c) $\boldsymbol{A}^+\boldsymbol{A} = (\boldsymbol{A}^+\boldsymbol{A})^T$
   d) $\boldsymbol{A}\boldsymbol{A}^+ = (\boldsymbol{A}\boldsymbol{A}^+)^T$
4. Let $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ with SVD $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$, and define
$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A}^T \\ \boldsymbol{A} & \boldsymbol{0} \end{bmatrix}.$$
   a) Express the eigenvalues of $\boldsymbol{H}$ in terms of the singular values of $\boldsymbol{A}$.
   b) Express the eigenvectors of $\boldsymbol{H}$ in terms of the singular vectors of $\boldsymbol{A}$.
   c) Extend above expressions to rectangular $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $m > n$
5. Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ denote image data from a work of artist A, and $\boldsymbol{B} \in \mathbb{R}^{m \times n}$ that from a work of artist B.
   a) Choose two images, and present a sequence of rank-one compressions of the image
   b) Construct a work using composition style of artist A (i.e., the large singular values and associated singular vectors) with the brush syle of artist B (i.e., the small singular values and associated singular vectors).
   c) Construct a work using composition style of artist B with the brush style of artist A.

   Note: The MATH662/images directory contains paintings