# Overview

- Intrinsic linear mapping representation
- SVD existence and uniqueness
- Reduced and full SVD

- $\boldsymbol{f}: U \to V$, $\boldsymbol{f}(a\boldsymbol{u}+b\boldsymbol{v}) = a\,f(\boldsymbol{u}) + b\,f(\boldsymbol{v})$, linear mapping
- $L = \{\boldsymbol{f} \mid \boldsymbol{f}: U \to V, \boldsymbol{f}(a\boldsymbol{u}+b\boldsymbol{v}) = a\,f(\boldsymbol{u}) + b\,f(\boldsymbol{v})\}$, set of linear mappings, itself a vector space, $\mathcal{L} = \{L, S, +, \cdot\}$
- $\boldsymbol{U} = [\, \boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \ldots \,]$ a basis for $U$, $U = C(\boldsymbol{U})$. Any $\boldsymbol{f} \in \mathcal{L}$ is represented by $\boldsymbol{A} = [\, \boldsymbol{f}(\boldsymbol{u}_1) \;\; \boldsymbol{f}(\boldsymbol{u}_2) \;\; \ldots \,]$
- Finite dimensional: $U = \mathbb{R}^n$, $V = \mathbb{R}^m \Rightarrow \boldsymbol{A} \in \mathbb{R}^{m \times n}$
- Infinite dimensional: $\int : \mathcal{C}^\infty \to \mathcal{C}^\infty \quad v(x) = \int u(x)\,\mathrm{d}x$ , $\boldsymbol{A} = \left[\, x \;\; \frac{1}{2}x^2 \;\; \ldots \,\right]$
- Example: $\sin(x) = \int \cos(x)\,\mathrm{d}x$, $\sin(x) = x - \frac{1}{3!}x^3 + \cdots$, $\cos(x) = 1 - \frac{1}{2!}x^2 + \cdots$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/3! \\ \vdots \end{bmatrix} \leftarrow \left[\, x \;\; \frac{1}{2}x^2 \;\; \frac{1}{3}x^3 \;\; \ldots \,\right] \begin{bmatrix} 1 \\ 0 \\ -1/2! \\ 0 \\ 1/4! \end{bmatrix}$$

# Intrinsic linear mapping representation

- Composition of linear mappings: $\boldsymbol{f} \leftrightarrow \boldsymbol{A}$, $\boldsymbol{g} \leftrightarrow \boldsymbol{B}$, $\boldsymbol{h} \leftrightarrow \boldsymbol{C}$, $\boldsymbol{h} = \boldsymbol{f} \circ \boldsymbol{g}$

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{C}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x})) = \boldsymbol{f}(\boldsymbol{B}\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{B}\boldsymbol{x} \Rightarrow \boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$$

- Decompose linear mapping $\boldsymbol{f}: U \to V$, $\boldsymbol{f} \leftrightarrow \boldsymbol{A}$, into simple transformations:
  - Linear mapping in basis $\boldsymbol{I}$: $\boldsymbol{I}\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{I}\boldsymbol{x}$
  - Change of bases $\boldsymbol{U}\boldsymbol{s} = \boldsymbol{I}\boldsymbol{y}$, $\boldsymbol{V}\boldsymbol{r} = \boldsymbol{I}\boldsymbol{x}$, $\boldsymbol{U}, \boldsymbol{V}$ orthogonal
  - Scaling $\boldsymbol{s} = \boldsymbol{\Sigma}\boldsymbol{r}$, $\boldsymbol{A} \in \mathbb{C}^{m \times n}$, $\boldsymbol{\Sigma} \in \mathbb{R}_{+}^{m \times n}$
  - Re-express mapping $\boldsymbol{U}\boldsymbol{s} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{r} = \boldsymbol{A}\boldsymbol{V}\boldsymbol{r} \Rightarrow \boldsymbol{U}\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{V} \Rightarrow \boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$

**Theorem.** *Every matrix* $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ *has a singular value decomposition (SVD)*

$$\boldsymbol{A} = \boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^*,$$

*with properties:*

1. $\boldsymbol{U} \in \mathbb{R}^{m \times m}$ *is an orthogonal matrix,* $\boldsymbol{U}^* \boldsymbol{U} = \boldsymbol{I}_m$;
2. $\boldsymbol{V} \in \mathbb{R}^{m \times m}$ *is an orthogonal matrix,* $\boldsymbol{V}^* \boldsymbol{V} = \boldsymbol{I}_n$;
3. $\boldsymbol{\Sigma} \in \mathbb{R}_+^{m \times n}$ *is diagonal,* $\boldsymbol{\Sigma} = \operatorname{diag}(\sigma_1, \ldots, \sigma_p)$, $p = \min(m, n)$, *and* $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_p \geqslant 0$.

# Reduced, full SVD

- $\boldsymbol{A} \in \mathbb{C}^{m \times n}$, $m \geqslant n$, full SVD

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^* = \begin{bmatrix} \boldsymbol{u}_1 & \ldots & \boldsymbol{u}_n & \boldsymbol{u}_{n+1} & \ldots & \boldsymbol{u}_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & & 0 \\ & & \vdots \\ & & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^* \\ \vdots \\ \boldsymbol{v}_n^* \end{bmatrix}$$

- Reduced SVD

$$\boldsymbol{A} = \hat{\boldsymbol{U}}\hat{\boldsymbol{\Sigma}}\boldsymbol{V}^* = \begin{bmatrix} \boldsymbol{u}_1 & \ldots & \boldsymbol{u}_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^* \\ \vdots \\ \boldsymbol{v}_n^* \end{bmatrix}$$