Overview

- SVD gives all matrix properties
- SVD low-rank approximations
- SVD computation overview

Matrix properties via the SVD

- Consider $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ with SVD $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^*$

```
octave> m=50; n=30; r=5; [U,R]=qr(rand(m)); [V,R]=qr(rand(n));
octave> S=zeros(m,n); sig=sort(3*rand(1,r),"descend");
octave> for i=1:r; S(i,i)=sig(i); end;
octave> A=U*S*V';
octave>
```

- $\operatorname{rank}(\boldsymbol{U}) = m$, $\operatorname{rank}(\boldsymbol{V}) = n$ (full rank)

  $\operatorname{rank}(\boldsymbol{A}) = \dim C(\boldsymbol{A}) = \dim C(\boldsymbol{A}^T) = r = \operatorname{rank}(\boldsymbol{\Sigma})$

```
octave> [rank(U) rank(V) rank(S) rank(A)]
```

$\left(50 \;\; 30 \;\; 5 \;\; 5 \right)$

```
octave>
```

Matrix properties via the SVD

- $C(\boldsymbol{A}) = \mathrm{span}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r) = \langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_r \rangle$, $N(\boldsymbol{A}) = \langle \boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_n \rangle$

```
octave> [rank(orth(A)) rank(null(A)) r min(m,n)-r]
```

(5 25 5 25)

```
octave>
```

- $\|\boldsymbol{A}\|_2 = \sigma_1$, $\|\boldsymbol{A}\|_F = \sigma_1^2 + \cdots + \sigma_r^2$

```
octave> [norm(A,2) sig(1)]
```

(2.786 2.786)

```
octave> [norm(A,"fro") norm(sig,2)]
```

(5.1424 5.1424)

```
octave>
```

SVD as a sum of rank-1 updates

- $\boldsymbol{A} = \sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T, r \leqslant \min(m, n).$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \dots & \boldsymbol{u}_r & \boldsymbol{u}_{r+1} & \dots & \boldsymbol{u}_m \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_r & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^T \\ \boldsymbol{v}_2^T \\ \vdots \\ \boldsymbol{v}_r^T \\ \vdots \\ \boldsymbol{v}_n^T \end{bmatrix} =$$

$$\begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \dots & \boldsymbol{u}_r & \boldsymbol{u}_{r+1} & \dots & \boldsymbol{u}_m \end{bmatrix} \begin{bmatrix} \sigma_1 \boldsymbol{v}_1^T \\ \sigma_2 \boldsymbol{v}_2^T \\ \vdots \\ \sigma_r \boldsymbol{v}_r^T \\ \vdots \\ 0 \end{bmatrix}$$

- Truncate sum of rank-1 updates

$$\boldsymbol{A} \cong \boldsymbol{A}_p = \sum_{i=1}^{p} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T.$$

- The truncation is an optimal approximation in the two norm

$$\boldsymbol{A}_p = \arg \inf_{\boldsymbol{B} \in \mathbb{C}^{m \times n}, \operatorname{rank}(\boldsymbol{B}) \leqslant p} \|\boldsymbol{A} - \boldsymbol{B}\|_2$$

- The error is known:

$$\|\boldsymbol{A} - \boldsymbol{A}_p\|_2 = \sigma_{p+1}, \|\boldsymbol{A} - \boldsymbol{A}_p\|_F = \sqrt{\sigma_{p+1}^2 + \cdots + +\sigma_r^2}$$

Image data

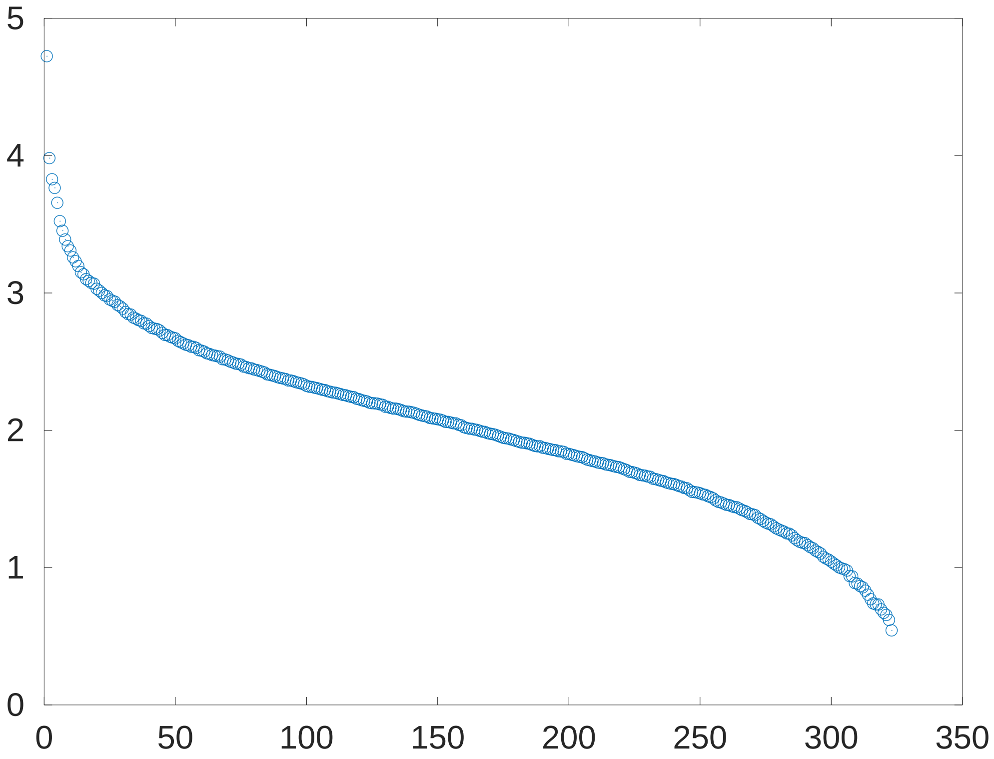
```
octave> cd ~/courses/MATH662/images;
        im=rgb2gray(imread("Frida_Kahlo_40.jpg"));
```

```
octave> figure(1); clf; imshow(im)
```

Image SVD

- Singular values often decay rapidly

```
octave> [U,S,V]=svd(im,1); figure(1); plot(log10(diag(S)),'o')
```



- Truncated rank-1 expansion can be used to compress images

```
octave> function im=svdim(S,U,V,p,px,py)
          im=S(1,1)*U(:,1)*V(:,1)';
          for k=2:p
            im=im+S(k,k)*U(1:px,k)*V(1:py,k)';
          end;
        end
```

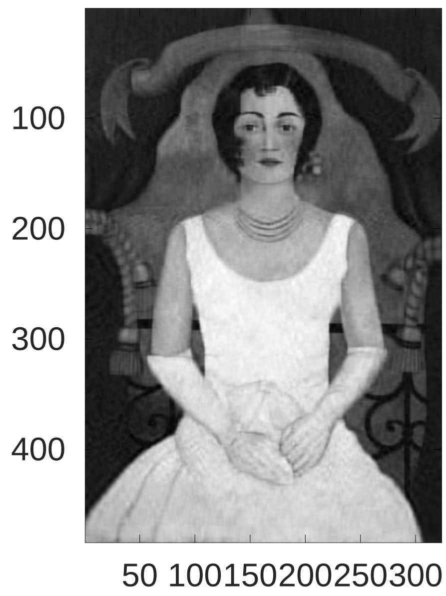
```
octave> [px py]=size(im); [rank(im) px py]
```

(323 484 323)

```
octave>
```

- In the example below $p = 50 \ll r = 323$

```
octave> figure(1); clf;
        imagesc(svdim(S,U,V,50,px,py)); colormap(gray); axis equal
```



SVD Computation Overview

- Compute $AA^* = (U\Sigma V^*)(U\Sigma V^*)^* = U\Sigma V^* V\Sigma^T U^* = U\Sigma\Sigma^T U^*$
- Compute $A^*A = (U\Sigma V^*)^*(U\Sigma V^*) = V\Sigma^T U^* U\Sigma V^* = V\Sigma^T\Sigma V^*$
- Recall matrix eigenrelationship $MX = X\Lambda$
- If $M = M^T$ then $\exists Q$ unitary (i.e., $QQ^* = Q^*Q = I$) s.t., $M = Q\Lambda Q^*$
- $\Sigma\Sigma^T, \Sigma^T\Sigma$ are diagonal, $\Sigma\Sigma^T = U^*(AA^*)U, \;\; \Sigma^T\Sigma = V^*(A^*A)V$
  - Squared singular values of A are eigenvalues of AA^* and A^*A
  - Left singular vectors U are eigenvectors of AA^*
  - Right singular vectors V are eigenvectors of A^*A