Overview

- Gram-Schmidt numerical experiments
- Householder reflectors

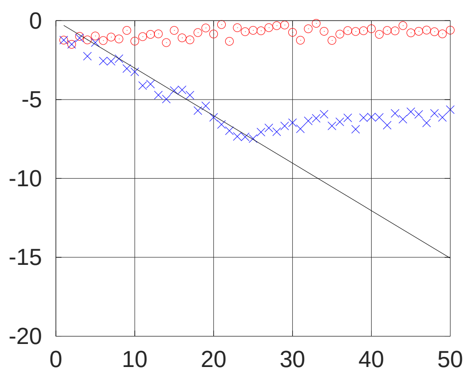
```
octave> function [Q,R] = cgs(A)
          [m,n]=size(A); Q=A; R=eye(n);
          for j=1:n
            v=A(:,j);
            for i=1:j-1
              R(i,j)=Q(:,i)'*A(:,j);
              v=v-R(i,j)*Q(:,i);
            end;
            R(j,j)=norm(v);
            if (R(j,j)<eps) break; endif;
            Q(:,j)=Q(:,j)/R(j,j);
          end;
        end
octave>
```

```
octave> function [Q,R] = mgs(A)
          [m,n]=size(A); Q=A; R=eye(n);
          for i=1:n
            R(i,i)=norm(Q(:,i));
            if (R(i,i)<eps) break; endif;
            Q(:,i)=Q(:,i)/R(i,i);
            for j=i+1:n
              R(i,j)=Q(:,i)'*A(:,j);
              Q(:,j)=Q(:,j)-R(i,j)*Q(:,i);
            end;
          end;
        end
octave>
```

```
octave> m=50; [U,X]=qr(randn(m)); [V,X]=qr(randn(m)); S=diag(2.^(-1:-1:-m)); A=U*S*V';
```

```
octave> [Qc,Rc]=cgs(A); [Qm,Rm]=mgs(A); function r=lgd(R); r=log10(diag(R)); end; i=1:m';
```

```
octave> figure(1); clf; plot(i,lgd(Rc),'or',i,lgd(Rm),'xb',i,lgd(S),'-k'); grid on
```



```
octave>
```

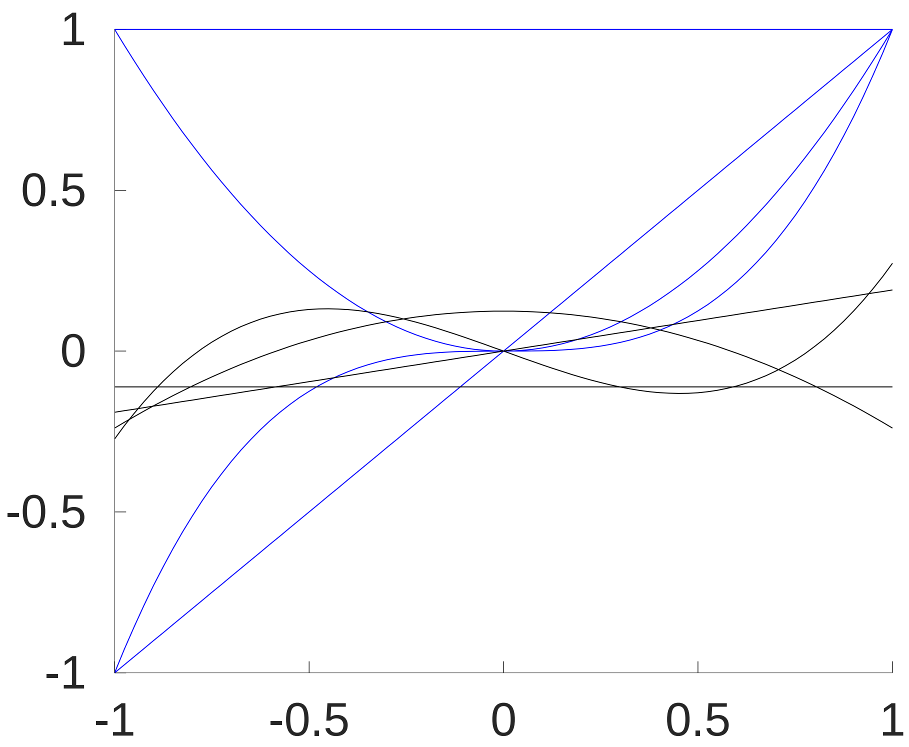
- Orthonormalization of monomial bases w.r.t. scalar products:

$(f,g)=\int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$ Legendre $(f,g)=\int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\,\mathrm{d}x$ Chebyshev

$(f,g)=\int_{0}^{\infty} e^{-x} f(x)g(x)\,\mathrm{d}x$ Laguerre $(f,g)=\int_{-\infty}^{\infty} e^{-x^2} f(x)g(x)\,\mathrm{d}x$ Hermite

```
octave> m=80; a=-1; b=1; h=(b-a)/m; x=(0:m)'*h+a;
        A=[x.^0 x.^1 x.^2 x.^3]; [Q,R]=qr(A);
```

```
octave> figure(1); clf; hold on;
        plot(x,A(:,1),'b',x,A(:,2),'b',x,A(:,3),'b',x,A(:,4),'b');
        plot(x,Q(:,1),'k',x,Q(:,2),'k',x,Q(:,3),'k',x,Q(:,4),'k')
```




```
octave>
```

- Orthonormalization of monomial bases w.r.t. scalar products:

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$$\boldsymbol{f}^T\boldsymbol{g}=\boldsymbol{f}^T\boldsymbol{I}^T\boldsymbol{I}\boldsymbol{g}=\boldsymbol{u}^T\boldsymbol{v}=\sum_{i=0}^{m}1\,f_i\,g_i,\;\boldsymbol{f}^T\boldsymbol{I}^T=\boldsymbol{u}^T\Rightarrow\boldsymbol{u}=\boldsymbol{I}\boldsymbol{f},\boldsymbol{v}=\boldsymbol{I}\boldsymbol{g}$$

$$(\boldsymbol{f},\boldsymbol{g})_{\boldsymbol{L}}=\boldsymbol{f}^T\boldsymbol{L}^T\boldsymbol{L}\boldsymbol{g},\|\boldsymbol{f}\|_{\boldsymbol{L}}=(\boldsymbol{f},\boldsymbol{f})=\boldsymbol{f}^T\boldsymbol{L}^T\boldsymbol{L}\boldsymbol{f}\geqslant 0$$

$$\boldsymbol{W}=\boldsymbol{Q}^T(\boldsymbol{\Sigma}^{1/2})^T\boldsymbol{\Sigma}^{1/2}\boldsymbol{Q}$$

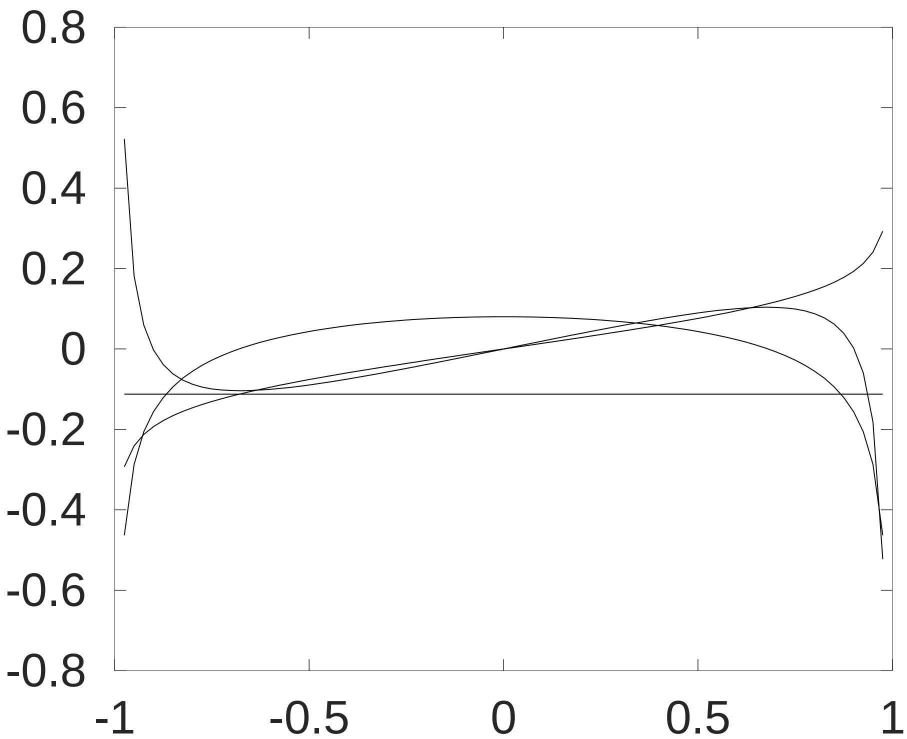
$$(\boldsymbol{f},\boldsymbol{g})_{\boldsymbol{W}}=\boldsymbol{f}^T\boldsymbol{W}\boldsymbol{g}=\boldsymbol{f}^T\boldsymbol{Q}^T(\boldsymbol{\Sigma}^{1/2})^T\boldsymbol{\Sigma}^{1/2}\boldsymbol{Q}\boldsymbol{g}$$

$$\boldsymbol{f}^T\boldsymbol{Q}^T(\boldsymbol{\Sigma}^{1/2})^T=\boldsymbol{u}^T\Rightarrow\boldsymbol{u}=\boldsymbol{\Sigma}^{1/2}\boldsymbol{Q}\boldsymbol{f},\boldsymbol{v}=\boldsymbol{\Sigma}^{1/2}\boldsymbol{Q}\boldsymbol{g}$$

$$\boldsymbol{u}^T\boldsymbol{v}=(\boldsymbol{f},\boldsymbol{g})_{\boldsymbol{W}}$$

```
octave> m=80; a=-1; b=1; h=(b-a)/m; x=(1:m-1)'*h+a;
        S=diag( (1-x.^2).^(-0.25) ); u=S*x;
        A=[u.^0 u.^1 u.^2 u.^3]; [Q,R]=qr(A); T=S*Q;
```

```
octave> figure(1); clf; hold off;
        plot(x,Q(:,1),'k',x,Q(:,2),'k',x,Q(:,3),'k',x,Q(:,4),'k')
```



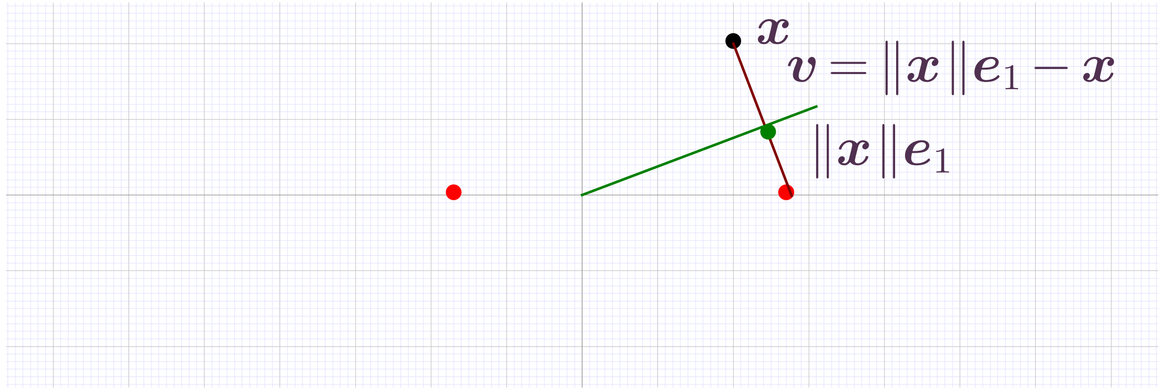

```
octave>
```

Householder reflector

- Gram-Schmidt: $\boldsymbol{A}\,\boldsymbol{R}_1 \ldots \boldsymbol{R}_r = \hat{\boldsymbol{Q}}$
- Householder triangularization $\boldsymbol{Q}_r \ldots \boldsymbol{Q}_1 \boldsymbol{A} = \boldsymbol{R}$, $\hat{\boldsymbol{Q}} = \boldsymbol{Q}_1^* \ldots \boldsymbol{Q}_r^*$, $\boldsymbol{A} = \hat{\boldsymbol{Q}}\,\boldsymbol{R}$

$$\boldsymbol{Q}_1 \begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix} = \begin{bmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{bmatrix}, \boldsymbol{Q}_2 \begin{bmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{bmatrix} = \begin{bmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & \star \end{bmatrix}$$

$$Py = \left(I - \frac{vv^*}{v^*v} \right) y$$



$$Py = y - 2\frac{vv^*}{v^*v} y$$

$$Hx = \|x\| \, e_1$$

$$H = I - \frac{2\,vv^*}{v^*v}$$

Householder QR factorization

$$\boldsymbol{Q}_k = \begin{bmatrix} \boldsymbol{I}_{k-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}_{m-k+1} \end{bmatrix}, \boldsymbol{Q}_k \boldsymbol{A}^{(k-1)} = \begin{bmatrix} \boldsymbol{I}_{k-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}_{m-k+1} \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_{k-1} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{D} \end{bmatrix}$$

$$\boldsymbol{Q}_k \boldsymbol{A}^{(k-1)} = \begin{bmatrix} \boldsymbol{I}_{k-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}_{m-k+1} \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_{k-1} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{k-1} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{H}_{m-k+1} \boldsymbol{D} \end{bmatrix}$$

$$\boldsymbol{x} = \boldsymbol{A}(k\!:m, k), \boldsymbol{v} = \|\boldsymbol{x}\| \, \boldsymbol{e}_1^{(m-k+1)} - \boldsymbol{x}$$

$$\boldsymbol{H}_{m-k+1} = \boldsymbol{I}_{m-k+1} - \frac{2\boldsymbol{v}\boldsymbol{v}^*}{\boldsymbol{v}^*\boldsymbol{v}}, \boldsymbol{H}_{m-k+1}\boldsymbol{a} = \boldsymbol{a} - \frac{2\boldsymbol{v}(\boldsymbol{v}^*\boldsymbol{a})}{\boldsymbol{v}^*\boldsymbol{v}}$$

$$\boldsymbol{Q}_n \ldots \boldsymbol{Q}_1 \boldsymbol{A} = \boldsymbol{R} \Rightarrow \boldsymbol{A} = (\boldsymbol{Q}_n \ldots \boldsymbol{Q}_1)^* \boldsymbol{R} = \boldsymbol{Q}_1^* \ldots \boldsymbol{Q}_n^* \boldsymbol{R} = \boldsymbol{Q}\boldsymbol{R}$$

$$\boldsymbol{Q}\boldsymbol{u} = \boldsymbol{Q}_1^* \ldots \boldsymbol{Q}_n^* \boldsymbol{u} = \begin{bmatrix} \boldsymbol{I}_{n-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{m-n+1} - \frac{2\boldsymbol{v}\boldsymbol{v}^*}{\boldsymbol{v}^*\boldsymbol{v}} \end{bmatrix} \ldots \left(\boldsymbol{u} - \frac{2\boldsymbol{v}(\boldsymbol{v}^*\boldsymbol{u})}{(\boldsymbol{v}^*\boldsymbol{v})} \right)$$

$$\text{FLOPs} = 3(m-n+1) + \cdots + 3(m-1) + 3m = 3n\left[m - \frac{(n-1)}{2}\right]$$