Overview

- Least squares:
  - normal equations
  - pseudo-inverse

Theorem. *Given* $\boldsymbol{A} \in \mathbb{C}^{m \times n}$, $\boldsymbol{b} \in \mathbb{C}^{m}$, $m \geqslant n$, $x \in \mathbb{C}^{n}$ *minimizes the norm of the residual* $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}$ *iff* $\boldsymbol{r} \perp C(\boldsymbol{A})$, *e.g.*

$$\boldsymbol{A}^{*}\boldsymbol{r} = 0 \Leftrightarrow \boldsymbol{A}^{*}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{b} \Leftrightarrow \boldsymbol{P}\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x},$$

with $\boldsymbol{P}$ *the orthogonal projector onto* $C(\boldsymbol{A})$. *The normal equation system*

$$\boldsymbol{A}^{*}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{b}$$

has $\boldsymbol{M} = \boldsymbol{A}^{*}\boldsymbol{A}$ *nonsingular iff* $\operatorname{rank}(\boldsymbol{A}) = n$.

Pseduo-inverse

- If $\operatorname{rank}(\boldsymbol{A}) = n$, then $\boldsymbol{x} = (\boldsymbol{A}^* \boldsymbol{A})^{-1} \boldsymbol{A}^* \boldsymbol{b} = \boldsymbol{A}^+ \boldsymbol{b}$ is the solution of the least squares problem, with $\boldsymbol{A}^+ = (\boldsymbol{A}^* \boldsymbol{A})^{-1} \boldsymbol{A}^*$ the *pseudo-inverse* of $\boldsymbol{A}$