Overview

- Least squares:
  - normal equations
  - pseudo-inverse
Theorem. Given $A \in \mathbb{C}^{m \times n}$, $b \in \mathbb{C}^m$, $m \geq n$, $x \in \mathbb{C}^n$ minimizes the norm of the residual $r = b - Ax$ iff $r \perp \mathcal{C}(A)$, e.g.

$$A^* r = 0 \iff A^* Ax = A^* b \iff Pb = Ax,$$

with $P$ the orthogonal projector onto $\mathcal{C}(A)$. The normal equation system

$$A^* Ax = A^* b$$

has $M = A^* A$ nonsingular iff $\text{rank}(A) = n$. 
If $\text{rank}(A) = n$, then $x = (A^*A)^{-1}A^*b = A^+b$ is the solution of the least squares problem, with $A^+ = (A^*A)^{-1}A^*$ the *pseudo-inverse* of $A$. 

Pseudo-inverse