Overview

- Conditioning:
  - problem condition number
  - of scaling, substraction
  - polynomial root finding (Wilkinson polynomial)

- A *problem* is some function between normed vector spaces $f: X \to Y$, with f typically continuous, nonlinear:
  - Find primitive $y(t) = \int x(t)\,\mathrm{d}t$, $f: \mathcal{L}_2(\mathbb{R}) \to \mathcal{L}_2(\mathbb{R})$, $y = f(x)$
  - Find derivative $y(t) = x'$, $f: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $y = f(x)$
  - For given basis set $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, find best linear approximant of $\boldsymbol{x} \in \mathbb{R}^m$, $\boldsymbol{y} = \arg\min_{\boldsymbol{u} \in \mathbb{R}^n} \|\boldsymbol{x} - \boldsymbol{A}\boldsymbol{u}\|$, $\boldsymbol{f}: \mathbb{R}^m \to \mathbb{R}^n$. $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$.
- All physical phenomena and floating point computations are affected by perturbations (quantum fluctuations, thermal background, floating point error)
  - A problem is *well conditioned* if $\delta f = f(x + \delta x) - f(x)$ small for δx small
  - A problem is *ill conditioned* if $\delta f = f(x + \delta x) - f(x)$ large for δx small
- Define absolute condition number $\hat{\kappa} = \lim_{\delta \to 0} \sup_{\|\delta x\|_X \leqslant \delta} \|\delta f\|_Y / \|\delta x\|_X$

- Relate perturbations to a reference quantity, define *relative condition number*

$$\kappa = \lim_{\delta \to 0} \sup_{\|\delta x\|_X \leqslant \delta} \frac{\|\delta f\|_Y}{\|f(x)\|_Y} \cdot \frac{\|x\|_X}{\|\delta x\|_X} = \sup_{\delta x} \frac{\|\delta f\|_Y}{\|f(x)\|_Y} \Big/ \frac{\|\delta x\|_X}{\|x\|_X}$$

- When f is differentiable (ordinary or Gateaux derivative) $\delta f = J \delta x$

$$\hat{\kappa} = \|J\|, \kappa = \frac{\|J\|}{\|f(x)\| / \|x\|}$$

- Examples:
  - $f: x \to x/2$ is differentiable, $J = 1/2$, $\kappa = 1$, well conditioned
  - $f: (x_1, x_2) \to x_1 - x_2$ is differentiable, $\boldsymbol{J} = [\ 1 \quad -1\ ]$. Use inf-norm

$$\kappa = \frac{\|J\|_\infty}{\|f(x)\|_\infty / \|x\|_\infty} = \frac{2 \max(|x_1|, |x_2|)}{|x_1 - x_2|} \to \infty \text{ for } x_1 \cong x_2$$

- Vieta relations $p_n(x) = \prod_{i=1}^{n}(x - r_i) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$

$$
\begin{aligned}
r_1 + r_2 + \cdots + r_n &= -a_{n-1} \\
r_1r_2 + r_1r_3 + r_1r_{n-1} + \cdots + r_{n-1}r_n &= a_{n-2} \\
r_1r_2r_3 + r_1r_2r_4 + \cdots + r_{n-2}r_{n-1}r_n &= -a_{n-3} \\
&\cdots \\
r_1r_2\ldots r_n &= (-1)^n a_0
\end{aligned}
$$

Wilkinson polynomial

- Let $p_n(x) = \prod_{i=1}^{n} (x - i) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Vietá relations:
  - $\sum_i i = -a_{n-1}$, $\sum_{i<j} i\,j = a_{n-2}$, $\sum_{i<j<k} i\,j\,k = -a_{n-3}$, ...
  - Define problem to find j^{th} root $f\colon \mathbb{R}^n \to \mathbb{C}$, $r_j = j = f(\boldsymbol{a})$
  - Condition number

$$\boldsymbol{J} = \left[\begin{array}{cccc} \frac{\partial r_j}{\partial a_0} & \frac{\partial r_j}{\partial a_1} & \ldots & \frac{\partial r_j}{\partial a_{n-1}} \end{array}\right] \Rightarrow \kappa = \frac{\|\boldsymbol{J}\|}{|r_j| / \|\boldsymbol{a}\|}$$

  - Consider perturbation of just δa_i

$$\kappa = \frac{|a_i\, r_j^{i-1}|}{|p'(r_j)|} \approx 10^{13} \text{ for } j = 15$$