Overview

- Conditioning of:
  - $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$
  - $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$
- Floating point axiom
- Stability

Condition of linear combination

- Consider problem $\boldsymbol{x} \to^{f} \boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{A} \in \mathbb{C}^{m \times n}$, $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$

$$\kappa = \sup_{\delta \boldsymbol{x}} \left(\frac{\|\delta \boldsymbol{f}\|}{\|\boldsymbol{f}\|} \Big/ \frac{\|\delta \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \right) = \sup_{\delta \boldsymbol{x}} \left(\frac{\|\boldsymbol{A}(\boldsymbol{x} + \delta \boldsymbol{x}) - \boldsymbol{A}\boldsymbol{x}\|}{\|\boldsymbol{A}\boldsymbol{x}\|} \Big/ \frac{\|\delta \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \right)$$

$$\kappa = \sup_{\delta \boldsymbol{x}} \left(\frac{\|\boldsymbol{A}\delta \boldsymbol{x}\|}{\|\boldsymbol{A}\boldsymbol{x}\|} \Big/ \frac{\|\delta \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \right) = \sup_{\delta \boldsymbol{x}} \left(\frac{\|\boldsymbol{A}\delta \boldsymbol{x}\|}{\|\delta \boldsymbol{x}\|} \right) \Big/ \frac{\|\boldsymbol{A}\boldsymbol{x}\|}{\|\boldsymbol{x}\|} = \|\boldsymbol{A}\| \frac{\|\boldsymbol{x}\|}{\|\boldsymbol{b}\|}$$

- $\boldsymbol{A} \in \mathbb{C}^{m \times m}, \operatorname{rank}(\boldsymbol{A}) = m \Rightarrow \|\boldsymbol{x}\| = \|\boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{x}\| \leqslant \|\boldsymbol{A}^{-1}\| \, \|\boldsymbol{A}\boldsymbol{x}\| \Rightarrow$

$$\frac{\|\boldsymbol{x}\|}{\|\boldsymbol{A}\boldsymbol{x}\|} = \frac{\|\boldsymbol{x}\|}{\|\boldsymbol{b}\|} \leqslant \|\boldsymbol{A}^{-1}\| \Rightarrow \kappa \leqslant \|\boldsymbol{A}\| \, \|\boldsymbol{A}^{-1}\| = \kappa(\boldsymbol{A})$$

- $\boldsymbol{A} \in \mathbb{C}^{m \times n} \Rightarrow \kappa \leqslant \|\boldsymbol{A}\| \, \|\boldsymbol{A}^{+}\| = \kappa(\boldsymbol{A})$

Matrix condition number, linear algebra condition numbers

- $\kappa(\boldsymbol{A}) = \|\boldsymbol{A}\| \, \|\boldsymbol{A}^+\|$ is the *condition number* of the matrix $\boldsymbol{A} \in \mathbb{C}^{m \times n}$
- $\boldsymbol{A} \in \mathbb{C}^{m \times m}, \operatorname{rank}(\boldsymbol{A}) = m \Rightarrow \kappa(\boldsymbol{A}) = \sigma_1 / \sigma_m$
- $\boldsymbol{A} \in \mathbb{C}^{m \times n}, \operatorname{rank}(\boldsymbol{A}) = n < m \Rightarrow \kappa(\boldsymbol{A}) = \sigma_1 / \sigma_n$
- $\boldsymbol{A} \in \mathbb{C}^{m \times n}, \operatorname{rank}(\boldsymbol{A}) < n < m \Rightarrow \kappa(\boldsymbol{A}) = \infty$
- Condition number of $\boldsymbol{x} \rightarrow \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\kappa = \|\boldsymbol{A}\| \, \|\boldsymbol{x}\| / \|\boldsymbol{b}\|$
- Condition number of $\boldsymbol{b} \rightarrow \boldsymbol{A}^{-1} \boldsymbol{b} = \boldsymbol{x}$, $\kappa = \|\boldsymbol{A}^{-1}\| \, \|\boldsymbol{b}\| / \|\boldsymbol{x}\|$
- Condition number of $\boldsymbol{A} \rightarrow \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$ ($\boldsymbol{b}$ constant)

$$(\boldsymbol{A} + \delta\boldsymbol{A})(\boldsymbol{x} + \delta\boldsymbol{x}) = \boldsymbol{b} \Rightarrow \delta\boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{A}\,\delta\boldsymbol{x} = 0 \Rightarrow \delta\boldsymbol{x} = -\boldsymbol{A}^{-1}\,\delta\boldsymbol{A}\,\boldsymbol{x} \Rightarrow$$

$$\frac{\|\delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leqslant \|\boldsymbol{A}^{-1}\| \, \|\delta\boldsymbol{A}\| \Rightarrow \kappa = \frac{\|\delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} / \frac{\|\delta\boldsymbol{A}\|}{\|\boldsymbol{A}\|} \leqslant \|\boldsymbol{A}^{-1}\| \, \|\boldsymbol{A}\| = \kappa(\boldsymbol{A})$$

- Equality can always be attained: $\kappa = \kappa(\boldsymbol{A})$

- $\mathbb{F}$ set of floating point numbers, $X \in \mathbb{R}$, $x = \mathrm{fl}(X) \in \mathbb{F}$
- Floating point operation, $x, y \in \mathbb{F}$, $x \circledast y = \mathrm{fl}(x * y)$, ($*$ operation in reals)

Axiom. *(Floating point)*$\forall x, y \in \mathbb{F}$, $\exists \epsilon$, $|\epsilon| \leqslant \epsilon_{\mathrm{mach}}$ *such that* $x \circledast y = (x * y)(1 + \epsilon)$

Accuracy

- Mathematical problem $f: X \to Y$
- Algorithm $\tilde{f}: X \to Y$
- Characterize perturbations due to approximation of the problem, relative error

$$\varepsilon = \frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|}$$

- Absolute error

$$e = \|\tilde{f}(x) - f(x)\|$$

- An algorithm is *accurate* if

$$\varepsilon = \frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = \mathcal{O}(\epsilon_{\text{mach}})$$

- Consider effect of pertubations of input $\tilde{x} = x + \delta x$ to algorithm

$$\frac{\|\tilde{x} - x\|}{\|x\|} = \mathcal{O}(\epsilon_{\text{mach}})\ \tilde{f}\ \text{forward stable if}\ \frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} = \mathcal{O}(\epsilon_{\text{mach}})$$

- $\tilde{f}$ is *backward stable*

$$\text{If}\ \exists \tilde{x}\ \frac{\|\tilde{x} - x\|}{\|x\|} = \mathcal{O}(\epsilon_{\text{mach}}),\ \tilde{f}(x) = f(\tilde{x})$$