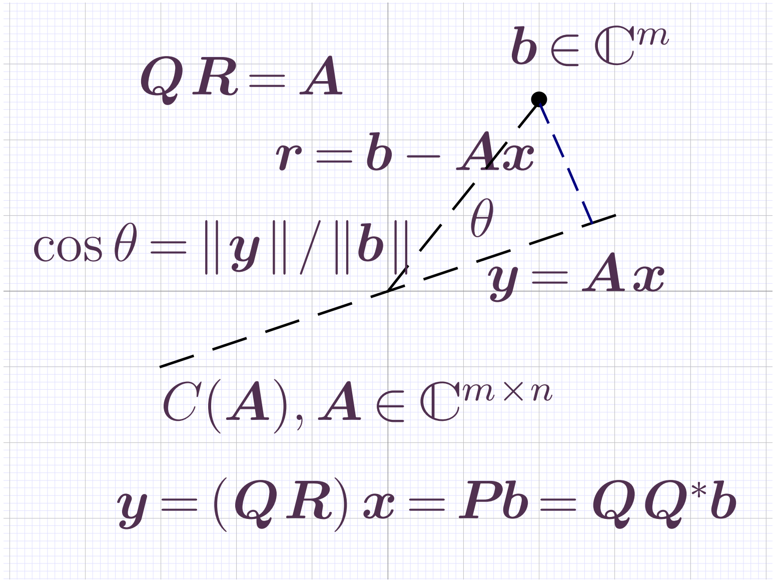
Overview

- Conditioning of least squares problem
- Stability of least squares algorithms

Consider normed spaces X, Y

Problem $f: X \to Y$	Algorithm $\tilde{f}: X \to Y$
Conditioning $\kappa = \sup_{\delta x} \frac{\lVert \delta f \rVert_Y}{\lVert f(x) \rVert_Y} / \frac{\lVert \delta x \rVert_X}{\lVert x \rVert_X}$	Relative error: $\varepsilon = \frac{\lVert \tilde{f}(x) - f(x) \rVert}{\lVert f(x) \rVert}$
if f differentiable $\kappa = \frac{\lVert J \rVert}{\lVert f(x) \rVert / \lVert x \rVert}, J = f'$	Absolute error: $e = \lVert \tilde{f}(x) - f(x) \rVert$
$\boldsymbol{x} \to^{f} \boldsymbol{A}\boldsymbol{x}$, $\kappa = \lVert \boldsymbol{A} \rVert \frac{\lVert \boldsymbol{x} \rVert}{\lVert \boldsymbol{b} \rVert} \leqslant \kappa(\boldsymbol{A}) = \lVert \boldsymbol{A} \rVert \, \lVert \boldsymbol{A}^{+} \rVert$	Accuracy: $\varepsilon = \mathcal{O}(\epsilon_{\text{mach}})$
$\boldsymbol{A} \to^{f} \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$, $\kappa = \kappa(\boldsymbol{A})$	Forward stability: $\lVert \tilde{x} - x \rVert / \lVert x \rVert = \mathcal{O}(\epsilon_{\text{mach}}) \Rightarrow$ $\lVert \tilde{f}(x) - f(\tilde{x}) \rVert / \lVert f(\tilde{x}) \rVert = \mathcal{O}(\epsilon_{\text{mach}})$
	Backward stability: $\tilde{f}(x) = f(\tilde{x}) \Rightarrow$ $\exists \tilde{x}, \lVert \tilde{x} - x \rVert / \lVert x \rVert = \mathcal{O}(\epsilon_{\text{mach}})$

in\out	$\boldsymbol{y}$	$\boldsymbol{x}$
$\boldsymbol{b}$	$\kappa = \dfrac{1}{\cos\theta}$	$\kappa = \dfrac{\kappa_{\boldsymbol{A}}}{\eta\cos\theta}$
$\boldsymbol{A}$	$\kappa = \dfrac{\kappa_{\boldsymbol{A}}}{\cos\theta}$	$\kappa = \kappa_{\boldsymbol{A}} + \dfrac{\kappa_{\boldsymbol{A}}^2 \tan\theta}{\eta}$



- $\boldsymbol{y} = \boldsymbol{P}\boldsymbol{b}, \boldsymbol{b} \to^{f} \boldsymbol{y}$, $\boldsymbol{f}(\boldsymbol{b}) = \boldsymbol{P}\boldsymbol{b}$, $\kappa = \|\boldsymbol{J}\|\,\|\boldsymbol{b}\| / \|\boldsymbol{y}\| = \|\boldsymbol{P}\|\,\|\boldsymbol{b}\| / \|\boldsymbol{y}\| = 1/\cos\theta$
- $\boldsymbol{x} = \boldsymbol{A}^{+}\boldsymbol{b}$, $\boldsymbol{b} \to^{g} \boldsymbol{x}$, $\boldsymbol{g}(\boldsymbol{b}) = \boldsymbol{A}^{+}\boldsymbol{b}$, $\kappa = \|\boldsymbol{A}^{+}\|\,\|\boldsymbol{b}\| / \|\boldsymbol{x}\| = \kappa(\boldsymbol{A}) / (\eta\cos\theta)$,

$$\eta = \frac{\|\boldsymbol{A}\|\,\|\boldsymbol{x}\|}{\|\boldsymbol{y}\|}$$

```
octave> m=100; n=15; t=(0:m-1)'/(m-1); A=[];
```

```
octave> for i=1:n
          A = [A t.^(i-1)];
        end
```

```
octave> b=exp(sin(4*t)); b=b/2006.787453080206;
```

```
octave> x=A\b; y=A*x; r=b-y; kappa=cond(A); [kappa]
```

22717770795.7495

```
octave> theta=asin(norm(r)/norm(b)); eta=norm(A)*norm(x)/norm(y);
```

```
octave> [Q,R]=qr(A,0); x=R\(Q'*b); x(15)
```

1

```
octave> x=A\b; x(15)
```

1

```
octave> [Q,R]=gs(A); x=R\(Q'*b); x(15)
```

```
'gs' undefined near line 1 column 7
```

```
octave> function [Q,R] = gs(A)
          [m,n]=size(A); Q=zeros(m,n); R=eye(n);
          for j=1:n
            v=A(:,j);
            for i=1:j-1
              R(i,j)=Q(:,i)'*A(:,j);
              v=v-R(i,j)*Q(:,i);
            end;
            R(j,j)=norm(v);
            if (R(j,j)<eps) break; endif
            Q(:,j)=v/R(j,j);
          end;
        end
```

```
parse error:

  syntax error

>>> function [Q,R] = gs(A)    [m,n]=size(A); Q=zeros(m,n); R=eye(n);    for
j=1:n      v=A(:,j);      for i=1:j-1        R(i,j)=Q(:,i)'*A(:,j);␣␣␣␣␣␣␣v=v-
R(i,j)*Q(:,i);␣␣␣␣␣end;␣␣␣␣␣R(j,j)=norm(v);␣␣␣␣␣if␣(R(j,j)<eps)␣break;␣endif␣␣␣␣␣

                                                                              ^
```

```
octave>
```