Overview

- LU factorization
- Forward substitution
- Backward substitution
- Pivoting
  - partial pivoting
  - full pivoting

- $\boldsymbol{b} \in \mathbb{C}^m$, m typically large. $\boldsymbol{I}\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{A} \in \mathbb{C}^{m\times m}$, $\mathrm{rank}(\boldsymbol{A}) = m$
- Recall QR-algorithm: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \Rightarrow (\boldsymbol{Q}\boldsymbol{R})\boldsymbol{x} = \boldsymbol{Q}(\boldsymbol{R}\boldsymbol{x}) = \boldsymbol{b} \Rightarrow \boldsymbol{R}\boldsymbol{x} = \boldsymbol{Q}^*\boldsymbol{b} = \boldsymbol{c}$
  1. $\boldsymbol{Q}\boldsymbol{R} = \boldsymbol{A}$, $\boldsymbol{Q}$ unitary, $\boldsymbol{R}$ upper triangular, $\mathcal{O}(m^3/3)$
  2. $\boldsymbol{c} = \boldsymbol{Q}^*\boldsymbol{b}$, $\mathcal{O}(m^2)$ flops
  3. $\boldsymbol{x} = \boldsymbol{R}^{-1}\boldsymbol{c} = \boldsymbol{R}\backslash\boldsymbol{c}$, $\mathcal{O}(m^2/2)$ flops
- Gaussian elimination: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \Rightarrow (\boldsymbol{L}\boldsymbol{U})\boldsymbol{x} = \boldsymbol{L}(\boldsymbol{U}\boldsymbol{x}) = \boldsymbol{b} \Rightarrow \boldsymbol{U}\boldsymbol{x} = \boldsymbol{L}^{-1}\boldsymbol{b} = \boldsymbol{c}$
  1. $\boldsymbol{L}\boldsymbol{U} = \boldsymbol{A}$, $\boldsymbol{L}$ lower triangular, $\boldsymbol{U}$ upper triangular, $\mathcal{O}(m^3/3)$
  2. $\boldsymbol{c} = \boldsymbol{L}\backslash\boldsymbol{b}$, forward substitution, $\mathcal{O}(m^2/2)$ flops
  3. $\boldsymbol{x} = \boldsymbol{U}\backslash\boldsymbol{c}$, backward substitution, $\mathcal{O}(m^2/2)$ flops

- Theory $\boldsymbol{L}_{m-1}\dots\boldsymbol{L}_1\boldsymbol{A}=\boldsymbol{U}\Rightarrow\boldsymbol{A}=\boldsymbol{L}_1^{-1}\dots\boldsymbol{L}_{m-1}^{-1}\boldsymbol{U}=\boldsymbol{L}\boldsymbol{U}$, $\boldsymbol{L}=\boldsymbol{L}_1^{-1}\dots\boldsymbol{L}_{m-1}^{-1}$
- Componentwise

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ -\frac{a_{12}}{a_{11}} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{1m}}{a_{11}} & 0 & \dots & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} & \dots & a_{1m} \\ a_{12} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{mm} \end{bmatrix}=\begin{bmatrix} a_{11} & a_{21} & \dots & a_{1m} \\ 0 & a_{22}' & \dots & a_{2m}' \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{2m}' & \cdots & a_{mm}' \end{bmatrix}$$

$$\begin{bmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{l}_1 & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} a_{11} & \boldsymbol{a}_{1,2:m}^T \\ \boldsymbol{a}_{2:m,1} & \boldsymbol{A}_1 \end{bmatrix}=\begin{bmatrix} a_{11} & \boldsymbol{a}_{1,2:m} \\ \boldsymbol{0} & \boldsymbol{l}_1\boldsymbol{a}_{1,2:m}^T+\boldsymbol{A}_1 \end{bmatrix}$$

- Componentwise algorithm

$$
\begin{array}{l}
\text{for } s=1 \text{ to } m-1 \\
\quad \text{for } i=s+1 \text{ to } m \\
\quad\quad \ell=-a_{si}/a_{ss} \\
\quad\quad \text{for } j=s+1 \text{ to } m \\
\quad\quad\quad a_{ij}=a_{ij}+\ell\, a_{sj}
\end{array}
$$

- Vector operation $\alpha\boldsymbol{x}+\boldsymbol{y}$, axpy, single precision: saxpy, double: daxpy

$$
\begin{array}{l}
\text{for } s=1 \text{ to } m-1 \\
\quad \text{for } i=s+1 \text{ to } m \\
\quad\quad \boldsymbol{a}_i^T=\boldsymbol{a}_i^T+(-a_{si}/a_{ss})\boldsymbol{a}_s^T=\mathrm{axpy}(-a_{si}/a_{ss},\boldsymbol{a}_s^T,\boldsymbol{a}_i^T)
\end{array}
$$

- Partial pivoting (within the current column)

for $i = 1$ to m $p_i = i$
for $s = 1$ to $m - 1$
  $\boldsymbol{p}$=pivot(s)
  for $i = s + 1$ to m
    $\ell = -a_{p(s)\,i} / a_{p(s)\,s}$
    for $j = s + 1$ to m
      $a_{p(i)\,j} = a_{p(i)\,j} + \ell\, a_{p(s)\,j}$