Homework 2

Due date: Feb 14, 2020, 11:55PM.

Bibliography: Trefethen & Bau, Lectures 10-14. Problems 1-4 = 1 pt each, Problem 5 = 4 points.

i. Exercise 10.1. Householder reflector $\boldsymbol{H}=\boldsymbol{I}-2\boldsymbol{v}\boldsymbol{v}^*$, $\boldsymbol{v}^*\boldsymbol{v}=1$, $\boldsymbol{v}\in\mathbb{C}^m$

a) Eigenvalues. $\boldsymbol{H}=\boldsymbol{H}^*\Rightarrow$ eigenvalues are real. $\boldsymbol{H}\boldsymbol{x}=\lambda\boldsymbol{x}\Rightarrow(\lambda-1)\boldsymbol{x}=2\boldsymbol{v}\boldsymbol{v}^*\boldsymbol{x}$, or $(\boldsymbol{v}\boldsymbol{v}^*)\boldsymbol{x}=\mu\boldsymbol{x}$, i.e., $\boldsymbol{H}$ has same eigenvectors as the rank-one projection matrix $\boldsymbol{P}_{\boldsymbol{v}}=\boldsymbol{v}\boldsymbol{v}^*$, and the eigenvalues are related by $\lambda=1-2\mu$. Since $\mu=0$ or $\mu=1$, $\lambda=1$ or $\lambda=-1$ (as expected for a unitary matrix). Eigenvectors of a reflector are either in the direction of $\boldsymbol{v}$ or orthogonal to $\boldsymbol{v}$. Reflector leaves unchanged vectors orthogonal to $\boldsymbol{v}$, switches orientation of those orthogonal to $\boldsymbol{v}$.

b) Determinant. $|\boldsymbol{H}|=\Pi_{i=1}^m\lambda_i=\pm1$.

c) SVD. $\boldsymbol{H}=(\ \boldsymbol{v}\ \ \boldsymbol{Q}\ )\,\mathrm{diag}(1,-1,-1,\ldots,-1)(\ \boldsymbol{v}\ \ \boldsymbol{Q}\ )^*$, with $\boldsymbol{Q}=(\ \boldsymbol{q}_1\ \ldots\ \boldsymbol{q}_{m-1})$, $\boldsymbol{q}_i^*\boldsymbol{v}=0$, $\boldsymbol{q}_i^*\boldsymbol{q}_j=\delta_{ij}$

ii. Exercises 10.2-3

10.2. QR factorization by Householder reflectors

Input: $\boldsymbol{A}\in\mathbb{R}^{m\times n}$
Output: $\boldsymbol{W}\in\mathbb{R}^{m\times r},\boldsymbol{R}\in\mathbb{R}^{r\times r}$
$k=1$; $\boldsymbol{W}=\mathrm{zeros}(m,n)$; $\boldsymbol{R}=\boldsymbol{A}$; $\mathrm{idx}=1{:}\,n$; $r=0$
for $j=1$ to n
  $\boldsymbol{x}=\boldsymbol{R}(j{:}\,m,\mathrm{idx}(j))$
  $\mathrm{nx}=\|\boldsymbol{x}\|$; $\boldsymbol{e}1=\boldsymbol{I}_{m-j+1}(:,1)$
  if $(\mathrm{nx}<\epsilon)$ then $\mathrm{idx}(j{:}\,n)=\mathrm{idx}(j{:}\,n)+1; \mathrm{idx}(n-r)=0$; loop
  $r=r+1$
  $\boldsymbol{v}=\mathrm{nx}\cdot\mathrm{sgn}(\boldsymbol{x}(1))\cdot\boldsymbol{e}1-\boldsymbol{x}$; $\mathrm{nv}=\|\boldsymbol{v}\|$
  if $(\mathrm{nv}>\epsilon)$ then
    $\boldsymbol{v}=\boldsymbol{v}/\mathrm{nv}$;
  $\boldsymbol{W}(j{:}\,m,k)=\boldsymbol{v}$; $k=k+1$
  $\boldsymbol{R}(j{:}\,m,\mathrm{idx}(j))=\mathrm{nx}\cdot\boldsymbol{e}1$
  for $l=j+1{:}\,n-r$
    $\boldsymbol{R}(j{:}\,m,\mathrm{idx}(l))=\boldsymbol{R}(j{:}\,m,\mathrm{idx}(l))-2\cdot\boldsymbol{v}\cdot(\boldsymbol{v}^*\cdot\boldsymbol{R}(j{:}\,m,\mathrm{idx}(l)))$
  end
end
return$(\boldsymbol{W}(:,1{:}\,r),\boldsymbol{R},r,\mathrm{idx})$

iii. Exercise 10.4

Algorithm 1

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Input: A [m x n]
Output: R [r x r] , W [m x r], idx [n], r

r = 0
idx = 1 : n
W = zeros( m, n, 'like', A );
R = A;
jq = 1;
for j = 1 : n
   x = R( j:m, idx( j ) )
   nx = norm( x, 2 )
   if ( nx < tol )
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        idx( j : n ) = idx( j : n ) + 1
        idx( n-r ) = 0
            continue
    end
    e1 = 0 * x;
    e1( 1 ) = 1;
    v = sgn( x( 1 ) ) * nx * e1 - x;
    nv = norm( v, 2 )
    if ( nv > tol )
        v = v / nv
    end
    W( j : m, j ) = v
    for l = j + 1 : n - r
        R( j : m, idx( l ) ) = R( j : m, idx( l ) ) - 2 * v * ( v' * R( j : m, idx( l ) ) )
    R( j : m, idx( j ) ) = nx * e1

    r += 1
    jq += 1
end
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iv. Exercises 11.1-2

v. Exercise 11.3