Test 3 Solution

Solve the following problems (6 course points each). Present a brief motivation of your method of solution. Explicitly state any conditions that must be met for solution procedure to be valid. No credit is awarded for statement of the final answer to a problem without presentation of solution procedure.

This is an open-book test, and you are free to consult the textbook or use software to inform your solution. Note however that the questions are so formulated that it is more efficient to draft the solution without use of software or consultation of the textbook; both of those actions would rapidly use up the allotted time. If you studied the course material and understood solutions to the homework assignments, drafting test question solutions in TeXmacs should take about 120 minutes, 60 minutes for Question 1, 60 minutes for Question 2. The allotted time is 3 hours, thus also providing flexibility for internet connection interruption.

Draft your solution in TeXmacs. At least 10 minutes before the submission cut-off time, upload your answer into Sakai.

Consider an initial-boundary-value problem for the diffusion partial differential equation $u_t = u_{xx}$, with conditions $u(t=0,x) = \sin x$, $u(t,x=0) = 0$, $u(t,x=\pi) = 0$, describing the cooling of a bar whose ends are maintained at constant temperature, $u: [0,\infty) \times [0,\pi] \to [0,\infty)$. The Crank-Nicolson numerical approximation is defined by $x_i = ih$, $i = 0, \ldots, m+1$, $h = \pi/(m+1)$, $t^n = nk$, $u_i^n \cong u(t^n, x_i)$

$$\frac{1}{k}(u_i^{n+1} - u_i^n) = \frac{1}{2}(\nabla_h^2 u_i^n + \nabla_h^2 u_i^{n+1}), \nabla_h^2 u_i^n = \frac{1}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n), \boldsymbol{u}^n = \begin{pmatrix} u_1^n & \ldots & u_m^n \end{pmatrix}^T. \quad (1)$$

1. Analyze the performance of Jacobi's method in solving (1) by carrying out the following steps:
   a) Determine the second-order differentiation matrix $\boldsymbol{D}$ that satistifies $h^2 \nabla_h^2 u_i^n = (\boldsymbol{D}\boldsymbol{u}^n)_i$, $i = 1, \ldots, m$.
   b) Using knowledge of the solution $(-\xi^2, e^{i\xi x})$ to the continuous eigenproblem $\partial_x^2 v = \lambda v$, find an analytical expression for the eigenvalues and eigenvectors of $\boldsymbol{D}$.
   c) Rewrite (1) as $\boldsymbol{A}\boldsymbol{u}^{n+1} = \boldsymbol{b}(\boldsymbol{u}^n)$. Express $\boldsymbol{A}, \boldsymbol{b}$ in terms of $\boldsymbol{I}, \boldsymbol{D}$ and $\sigma = k/(2h^2)$.
   d) Determine the spectral radius of $\boldsymbol{A}$, $\rho(\boldsymbol{A})$.
   e) Determine the spectral radius of $\boldsymbol{M}$, $\rho(\boldsymbol{M})$ where $\boldsymbol{M}$ is the matrix arising from the Jacobi iteration

$$\boldsymbol{A} = \boldsymbol{L} + \boldsymbol{C} + \boldsymbol{U}, \boldsymbol{M} = -\boldsymbol{C}^{-1}(\boldsymbol{L} + \boldsymbol{U}).$$

   f) Estimate the number N of Jacobi iterations required to reduce the absolute error $e_k = \|\boldsymbol{u}_{k+1}^n - \boldsymbol{u}_k^n\|$ from inital value e_0 to some desired $e_N = \varepsilon$.

Solution. In addition to the theoretical derivations, the solution shows how small Octave computations can be used as a verification.

a) Since boundary conditions imply $u_0 = u_{m+1} = 0$, $\boldsymbol{D} = \operatorname{diag}([\,1 \;\; -2 \;\; 1\,])$, i.e.,

$$\boldsymbol{D} = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}.$$

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octave] m=8; h=pi/(m+1); o=ones(m-1,1); D = diag(o,-1)-2*eye(m) + diag(o,1)
D =

  -2   1   0   0   0   0   0   0
   1  -2   1   0   0   0   0   0
   0   1  -2   1   0   0   0   0
   0   0   1  -2   1   0   0   0
   0   0   0   1  -2   1   0   0
   0   0   0   0   1  -2   1   0
   0   0   0   0   0   1  -2   1
   0   0   0   0   0   0   1  -2

octave]
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b) Restriction of the eigenproblem $\partial_x^2 v = \lambda v$ to $x \in [0,\pi]$ with boundary conditions $v(0) = v(\pi) = 0$, leads to solution $(-l^2, \sin lx)$ with $l \in \mathbb{Z}$. Discretization of the continuum eigenfunction solution suggests eigenvector with components $r_j = \sin(jlh)$, and j^{th} component of eigenproblem $\boldsymbol{D}\boldsymbol{r} = \lambda \boldsymbol{r}$ is

$$r_{j-1} - 2r_j + r_{j+1} = \lambda r_j \Rightarrow \sin[(j-1)\,lh] - 2\sin(jlh) + \sin[(j+1)\,lh] = \lambda \sin(jlh).$$

Use trigonometric identity $\sin\alpha + \sin\beta = 2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\beta-\alpha)\right]$ to obtain

$$\lambda = 2[\cos(lh) - 1] = -4\sin^2\frac{lh}{2} = -l^2 h^2 + \mathcal{O}(h^4),$$