Test 1

Solve the following problems (4 course points each). Present a brief motivation of your method of solution.

1. Let $\boldsymbol{A}^+(x)$ denote the pseudoinverse of $\boldsymbol{A}(x) \in \mathbb{R}^{2\times 2}$ defined as

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}, x \in \mathbb{R}.$$

a) Is $\boldsymbol{A}^+(x)$ continuous at $x=0$?

Solution. The SVD of $\boldsymbol{A}(x) = \boldsymbol{U\Sigma V}^T$ differs for $0 \leqslant x < 1$, and $-1 < x < 0$

$$0<x<1 \quad \boldsymbol{A}_+(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$-1<x<0 \quad \boldsymbol{A}_-(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -x \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The corresponding pseudoinverses $\boldsymbol{A}^+(x) = \boldsymbol{V\Sigma}^+\boldsymbol{U}^T$ are

$$0<x<1 \quad \boldsymbol{A}_+^+(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & x^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & x^{-1} \end{bmatrix},$$
$$x=0 \quad \boldsymbol{A}_0^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$-1<x<0 \quad \boldsymbol{A}_-^+(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -x^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & x^{-1} \end{bmatrix}.$$

Note that

$$\lim_{x\to 0, x>0} \boldsymbol{A}_+^+(x) = \begin{bmatrix} 1 & 0 \\ 0 & \infty \end{bmatrix} \neq \lim_{x\to 0, x<0} \boldsymbol{A}_-^+(x) = \begin{bmatrix} 1 & 0 \\ 0 & -\infty \end{bmatrix} \neq \boldsymbol{A}_0^+(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

hence $\boldsymbol{A}^+(x)$ is not continuous at $x=0$.

b) Estimate the error $e = \|\boldsymbol{A}^+(\varepsilon) - \boldsymbol{A}^+(0)\|_2$ for small ε.

Solution. Using above expressions,

$$e = \left\| \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix} \right\|_2,$$

and by properties of the SVD $e = \varepsilon^{-1}$.

c) Comment on floating-point computational implications of your results from (a) and (b).

Solution. The discontinuity of $\boldsymbol{A}^+(x)$ indicates unbounded amplification of floating point representation errors in expressions such $\boldsymbol{x} = \boldsymbol{A}^+\boldsymbol{b}$ to compute the (generalized) solution of $\boldsymbol{Ax} = \boldsymbol{b}$, and more stable numerical procedures are needed than the explicit computation of the pseudoinverse.

2. Find the singular value decomposition of

$$\boldsymbol{A} = \begin{bmatrix} -4 & -6 \\ 3 & -8 \end{bmatrix},$$

showing all intermediate steps.

Solution. From $\boldsymbol{A} = \boldsymbol{U\Sigma V}^T$, deduce that $\boldsymbol{AA}^T = \boldsymbol{U\Sigma}^2\boldsymbol{U}^T$, and $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{V}^T\boldsymbol{\Sigma}^2\boldsymbol{V}$

$$\boldsymbol{AA}^T = \begin{bmatrix} -4 & -6 \\ 3 & -8 \end{bmatrix}\begin{bmatrix} -4 & 3 \\ -6 & -8 \end{bmatrix} = \begin{bmatrix} 52 & 36 \\ 36 & 73 \end{bmatrix},$$

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{bmatrix} -4 & 3 \\ -6 & -8 \end{bmatrix}\begin{bmatrix} -4 & -6 \\ 3 & -8 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 100 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 100 & 0 \\ 0 & 25 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Deduce that the singular values are $\sigma_1=10$, $\sigma_2=5$, and the singular vectors are

$$\boldsymbol{V}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \boldsymbol{U}=\boldsymbol{A}\boldsymbol{V}\boldsymbol{\Sigma}^{-1}=\begin{bmatrix} -4 & -6 \\ 3 & -8 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/10 & 0 \\ 0 & 1/5 \end{bmatrix}=\begin{bmatrix} -6 & -4 \\ -8 & 3 \end{bmatrix}\begin{bmatrix} 1/10 & 0 \\ 0 & 1/5 \end{bmatrix}=\begin{bmatrix} -.6 & -.8 \\ -.8 & .6 \end{bmatrix}.$$

3. Let $\boldsymbol{A}^+$ denote the pseudoinverse of $\boldsymbol{A}\in\mathbb{C}^{m\times n}$. Express the four fundamental vector subspaces of $\boldsymbol{A}^+$ in terms of those of $\boldsymbol{A}$.

Solution. Let $r=\operatorname{rank}(\boldsymbol{A})$ and write the block form of the SVD of $\boldsymbol{A}$ and $\boldsymbol{A}^+$

$$\boldsymbol{A}=[\,\boldsymbol{U}_1\ \ \boldsymbol{U}_2\,]\boldsymbol{\Sigma}\begin{bmatrix} \boldsymbol{V}_1^* \\ \boldsymbol{V}_2^* \end{bmatrix}, \boldsymbol{A}^+=\boldsymbol{V}\boldsymbol{\Sigma}^+\boldsymbol{U}^T=[\,\boldsymbol{V}_1\ \ \boldsymbol{V}_2\,]\boldsymbol{\Sigma}^+\begin{bmatrix} \boldsymbol{U}_1^* \\ \boldsymbol{U}_2^* \end{bmatrix}.$$

with $\boldsymbol{U}_1\in\mathbb{C}^{m\times r}, \boldsymbol{U}_2\in\mathbb{C}^{m\times(m-r)}$, $\boldsymbol{V}_1\in\mathbb{C}^{n\times r}$, $\boldsymbol{V}_2\in\mathbb{C}^{n\times(n-r)}$. The four fundamental spaces of $\boldsymbol{A}$ are

$$C(\boldsymbol{A})=C(\boldsymbol{U}_1), N(\boldsymbol{A}^*)=C(\boldsymbol{U}_2), C(\boldsymbol{A}^*)=C(\boldsymbol{V}_1), N(\boldsymbol{A})=C(\boldsymbol{V}_2).$$

The four fundamental spaces of $\boldsymbol{A}^+$ are

$$C(\boldsymbol{A}^+)=C(\boldsymbol{V}_1), N(\boldsymbol{A}^{+*})=C(\boldsymbol{V}_2), C(\boldsymbol{A}^{+*})=C(\boldsymbol{U}_1^*), N(\boldsymbol{A}^+)=C(\boldsymbol{U}_2^*).$$

Deduce that

$$C(\boldsymbol{A}^+)=C(\boldsymbol{A}^*), N(\boldsymbol{A}^{+*})=N(\boldsymbol{A}), C(\boldsymbol{A}^{+*})=C(\boldsymbol{A}), N(\boldsymbol{A}^+)=N(\boldsymbol{A}^*).$$