Test 2

Solve the following problems (6 course points each). Present a brief motivation of your method of solution.

1. Consider a computer satisfying the floating point arithmetic axiom $x \circledast y = (x * y)(1+\epsilon)$ for all $x, y \in \mathbb{F} \subset \mathbb{R}$ (set of real floating point numbers), with machine epsilon denoted by ϵ, $\circledast$ a floating point operation, $*$ the corresponding real number operation. Also consider construction of the Newton interpolating poynomial

$$p_n(x) = y_0 + [y_1, y_0](x - x_0) + \cdots + [y_n, y_{n-1}, \ldots, y_0](x - x_0)\ldots(x - x_{n-1}) = a_0 + a_1 x + \cdots + a_n x^n$$

of data $\mathcal{D} = \{(x_k, y_k), x_k = kh, k = 0, \ldots, n\}$, $h = 1/n$, $n \in \mathbb{N}_+$, with divided differences defined by $[y_k] = y_k$,

$$[y_k, y_{k-1}, \ldots, y_{k-l}] = \frac{[y_k, y_{k-1}, \ldots, y_{k-l+1}] - [y_{k-1}, y_{k-2}, \ldots, y_{k-l}]}{x_k - x_{k-l}} = \frac{[y_k, y_{k-1}, \ldots, y_{k-l+1}] - [y_{k-1}, y_{k-2}, \ldots, y_{k-l}]}{lh}.$$

   a) What is the condition number of the problem $\mathcal{D} \rightarrow^f a_n$?
   b) Estimate the error δa_n produced by error δy_j in j^{th} data measurement, i.e., $\tilde{y_j} = y_j + \delta y_j$.
   c) What is the condition number of the problem $h \rightarrow^g a_n$?
   d) Is the evaluation of $g(h) = a_n(h)$ well-conditioned, ill-conditioned or ill-posed? Consider limiting values of the sampling step size h.

2. Let $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ denote the matrix obtained by second-order accurate, centered finite difference approximation of the Helmholtz equation $\nabla^2 u = -k^2 u$ in $(0,1) \times (0,1)$ with periodic boundary conditions $u(x+p, y+q) = u(x,y)$, $p, q \in \mathbb{Z}$, $h = 1/n$, $n \in \mathbb{N}_+$, $m = n^2$, leading to the eigenvalue problem $\boldsymbol{A}\boldsymbol{u} = -h^2 k^2 \boldsymbol{u}$.

$$(\nabla^2 u)_{ij} \cong \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2} \Rightarrow \boldsymbol{A} = \operatorname{diag}([0, \ldots, 1, \ldots, 1, -4, 1, \ldots, 1, \ldots 0])$$

   a) Present an algorithm to reduce $\boldsymbol{A}$ to symmetric Hessenberg form $\boldsymbol{H}$ using Householder reflectors that preserves eigenvalues of $\boldsymbol{A}$.
   b) Is the algorithm accurate in floating point arithmetic?
   c) Is the algorithm forward stable?
   d) Is the algorithm backward stable?

   Provide either an analysis or a qualitative motivation using established theorems for answers to (b)-(d).