MATH662: Numerical linear algebra April 29, 2021

Test 3

Implement and carry out numerical experiments on the following algorithm. Comment on the observed numerical behavior (all aspects: computational complexity, accuracy, stability) using theoretical concepts learned throughout the course.

Let $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ be a dense symmetric matrix. Jacobi's method to find eigenvalues sets $\boldsymbol{A}_0 = \boldsymbol{A}$, and constructs a sequence of matrices $\{\boldsymbol{A}_n\}_{n \in \mathbb{N}}$ through rotation similarity transforms

$$\boldsymbol{A}_m = \boldsymbol{J}_{m-1}^T \boldsymbol{A}_{m-1} \boldsymbol{J}_{m-1} = \boldsymbol{J}_{m-1}^T \boldsymbol{J}_{m-2}^T \boldsymbol{A}_{m-2} \boldsymbol{J}_{m-2} \boldsymbol{J}_{m-1} = \cdots = \boldsymbol{J}^T \boldsymbol{A}_0 \boldsymbol{J}.$$

The rotation matrix that nullifies entry at position (i,j) in step k of the algorithm is

$$\boldsymbol{J}_k = \boldsymbol{R}(i,j,\theta) = \begin{array}{cc} & \begin{array}{cccccccc} & & & i & & j & & \end{array} \\ \begin{array}{c} \\ \\ \\ i \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & \cos\theta & & -\sin\theta & & \\ & & & & \ddots & & & \\ & & & \sin\theta & & \cos\theta & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \end{array}.$$

Gray area is used to indicate row, column positions and is not part of the $\boldsymbol{J}_k$ matrix. At each iteration k, (i,j) are chosen to correspond to the largest (in absolute value) off-diagonal element of $\boldsymbol{A}_k$.

1. Write the algorithm in pseudo-code (3 points).
2. Implement the algorithm in language of your choice (Octave is easiest and preferred, within this TeXmacs document to enable commenting even more preferred) (3 points).
3. Test the algorithm on a matrix of your choice. Use a reasonable size, i.e., neither $m=3$, nor $m=10^6$ are "reasonable" to assess the algorithm. Plot the 2-norm of the off-diagonal entries at each iteration as a function of iteration number. (3 points).
4. Apply theoretical concepts to assess the efficacy of the algorithm. For example: do you expect the algorithm to generate a diagonal matrix in some finite number of steps N?; if not, would you expect it to converge to a diagonal matrix? What would the effect of disparate or, alternatively many repeated, eigenvalues be?