Test 2

Solve the following problems (6 course points each). Present a brief motivation of your method of solution.

1. Consider the variant of Horner's algorithm for evaluation of a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ that saves intermediate results in $(p_n, p_{n-1}, \ldots, p_0) \in \mathbb{R}^{n+1}$, with final result returned in p_0

```
Input: x, a_n, ..., a_0
p_n = a_n
for i = n-1 downto 0
   p_i = x · p_{i+1} + a_i
end
return(p_0, ..., p_n)
```

Admiting the floating point arithmetic axiom $x \circledast y = (x * y)(1+\epsilon)$ for all $x, y \in \mathbb{F}$ (set of real floating point numbers), find an expression for $\delta p = p(x+\delta x) - p(x)$ with δx denoting the ensemble of floating point errors introduced by every arithmetic operation in Horner's algorithm

a) Is polynomial evaluation backward stable?

b) Show that the relative condition number of polynomial evaluation is well approximated by

$$\kappa = \frac{|a_0| + |a_1 x| + \cdots + |a_n x^n|}{|a_0 + a_1 x + \cdots + a_n x^n|}$$

c) Is the evaluation of $z(x) = 0$ well-conditioned, ill-conditioned or ill-posed?

d) Define the relative distance between two polynomials p, q with coefficients $a = (a_0, a_1, \ldots, a_n)$, $b = (b_0, b_1, \ldots, b_n)$ as

$$d(p,q) = \max_i \left| \frac{a_i - b_i}{a_i} \right|.$$

Show that the inverse condition number of evaluating p is also the minimal distance to an ill-posed polynomial evaluation.

Solution. Assuming that a machine epsilon floating point error ϵ is introduced by each operation, at stage i

$$\tilde{p}_i = (x \cdot \tilde{p}_{i+1}(1+\epsilon) + a_i)(1+\epsilon) \Rightarrow \tilde{p}_i = x \cdot \omega^2 \cdot \tilde{p}_{i+1} + \omega \cdot a_i = y \cdot \tilde{p}_{i+1} + b_i,$$

with $y = \omega^2 x$, $b_i = \omega \cdot a_i$ ($b_n = \omega \cdot a_n$ is also included to force symmetric treatment of polynomial coefficients, and can be considered the floating point result of $1 \cdot a_n$). The effect of floating point errors in Horner's algorithm is to evaluate the polynomial

$$\tilde{q}(y) = b_n y^n + b_{n-1} y^{n-1} + \cdots + b_0.$$

Substitute to obtain expression in terms of $x, a_0, \ldots, a_n$

$$\tilde{q}(y) = \tilde{q}(\omega^2 x) = \tilde{p}(x) = \omega[\omega^{2n} a_n x^n + \omega^{2(n-1)} a_{n-1} x^{n-1} + \ldots + \omega^2 a_1 x + a_0],$$

and obtain the polynomial approximant

$$\tilde{p}(x) = \tilde{a}_n x^n + \tilde{a}_{n-1} x^{n-1} + \ldots + \tilde{a}_1 x + \tilde{a}_0$$

with coefficients $\tilde{a}_i = \omega^{2i+1} a_i \cong [1 + (2i+1)\epsilon] a_i$.

a) Backward stability requires $\tilde{p}(x) = p(\tilde{x})$ for some $\tilde{x}$ close to x, $|\tilde{x} - x| / |x| = \mathcal{O}(\epsilon)$. When considering $\tilde{p}(x), p(\tilde{x}) \in \mathbb{F}$, note that $\tilde{p}(x) = \omega p(\tilde{x})$ for $\tilde{x} = \omega^2 x \cong (1+2\epsilon)x$, to within machine precision (recall that $\mathrm{fl}(x) = x(1+\epsilon)$), hence Horner's scheme is backward stable.

Alternatively, if $\tilde{p}(x) = p(\tilde{x})$ is viewed as an equality between reals, $\tilde{p}(x), p(\tilde{x}) \in \mathbb{R}$, solving the equation

$$f(\tilde{x}) = p(\tilde{x}) - \tilde{p}(x) = p(\tilde{x}) - \omega p(\omega^2 x) = 0, \qquad (1)$$

gives $\tilde{x}(x;\omega,a)$, and an estimate of $|\tilde{x}-x|$ is needed. Let $x_1, \ldots, x_n$ be the roots of $p(x)=0$. The solutions to (1) arise from perturbation of the coefficient a_0 of p, $a_0 \to a_0 - \omega p(\omega^2 x)$. This affects the n^{th} Vieta relation which for roots $x_1, \ldots, x_n$ of $p(x)$ is

$$x_1 x_2 \ldots x_n = (-1^n) a_0 / a_n,$$

while for roots $\tilde{x}_1, \ldots, \tilde{x}_n$ of $p(\tilde{x}) - \omega p(\omega^2 x)$ is

$$\tilde{x}_1 \tilde{x}_2 \ldots \tilde{x}_n = (-1^n)[a_0 - \omega p(\omega^2 x)] / a_n = x_1 x_2 \ldots x_n - (-1^n)\, \omega\, p(\omega^2 x) / a_n,$$

b) Since $p(x)$ is differentiable the relative condition number is

$$\kappa = \frac{|p'(x)|}{|p(x)|} |x|$$

Consider Taylor series $p(0) = p(x) + p'(\xi)(0-x)$, with $|\xi| < |x|$, whence $x p'(\xi) = p(x) - p(0)$, and

$$|x p'(\xi)| = |x|\, |p'(\xi)| = |p(x) - p(0)| = \left| \sum_{i=0}^{n} a_i x^i - a_0 \right| = \left| \sum_{i=1}^{n} a_i x^i \right| \leqslant \sum_{i=1}^{n} |a_i x^i| \leqslant \sum_{i=0}^{n} |a_i x^i|$$

Obtain

$$\kappa = \frac{|p'(x)|}{|p(x)|} |x| \leqslant \frac{\sum_{i=0}^{n} |a_i x^i|}{|\sum_{i=1}^{n} a_i x^i|},$$

to within the error of the Taylor series $\mathcal{O}(|x|)$.

Alternatively, use

$$\kappa_r = \lim_{\delta \to 0} \sup_{\|\delta x\| \leqslant \delta} \frac{|\delta p|\, |x|}{|p|\, |\delta x|}, \tag{2}$$

and valuate

$$|\delta p| = |\tilde{p}(x) - p(x)| = \left| \sum_{i=0}^{n} (\tilde{a}_i - a_i) x^i \right| = \left| \sum_{i=0}^{n} i \epsilon a_i x^i \right| \leqslant \left| n\epsilon \sum_{i=0}^{n} a_i x^i \right| \leqslant n\epsilon \sum_{i=0}^{n} |a_i x^i|,$$

and replace in (1) to obtain

$$\kappa_r \leqslant \lim_{\delta \to 0} \sup_{\|\delta x\| \leqslant \delta} \frac{\sum_{i=0}^{n} |a_i x^i|}{|\sum_{i=0}^{n} a_i x^i|} \cdot \frac{n\epsilon\, |x|}{|\delta x|} \cong \frac{\sum_{i=0}^{n} |a_i x^i|}{|\sum_{i=0}^{n} a_i x^i|} n\, |x| = \kappa n\, |x|.$$

c) The absolute condition number for the problem $x \to 0$ is

$$\hat{\kappa} = \lim_{\delta \to 0} \sup_{\|\delta x\| \leqslant \delta} \frac{\|\delta z\|}{\|\delta x\|} = \lim_{\delta \to 0} \sup_{\|\delta x\| \leqslant \delta} \frac{0}{\|\delta x\|}$$

For $z(x) = 0$, the polynomial coefficients are $z_i = 0 \in \mathbb{R}$ with floating point representation $a_i = \mathrm{fl}(0) \leqslant \epsilon$. The degree of $z(x)$ is zero and worst case evaluation by Horner's scheme would return $\tilde{z}(x) = a_n = \epsilon$. The absolute condition number is $\kappa = \epsilon / 0 = \infty$, hence the problem is ill-posed.

d) Since $z(x) = 0$ is ill-posed

$$d(p, z) = \kappa =$$

2. Write an algorithm that uses Householder reflectors to reduce a symmetric matrix $A \in \mathbb{R}^{m \times m}$ to tridiagonal form.

Solution.