MATH761 Oct 15, 2018

Midterm Examination

Instructions:

- This is a take-home examination due on Wednesday, Oct. 17, 2018 at 5:00PM. You are free to use library, software, and online resources, but all work is individual with no discussion of examination topics with other students or faculty. Submission of an answer is an implicit honor pledge to have respected the above rules.
- Provide your answer by completing the present TeXmacs file, and submitting through Sakai. Only one submission is allowed, with the above deadline, hence be ready ahead of time.
- An answer should require about 3 hours to read and summarize the background material and 3 hours of work to answer the subsequent questions if course topics and previous graduate study material has been well understood. Questions are meant to be answered by analytical computations, not numerical simulation. Feel free to use *Mathematica* and illustrate your answers by plots.

1. Read sections 1-3 of the paper "Group velocity in finite difference schemes", published in SIAM Review, Vol. 24, No. 2, April 1982, authored by Lloyd N. Trefethen, available at https://people.maths.ox.ac.uk/trefethen/publication/PDF/1982_6.pdf. Present a three-paragraph summary of the main points of these paper sections.

2. Consider the longitudinal plane wave $\boldsymbol{u}_i(t,x)=\boldsymbol{e}_x\exp[i(\omega t-\xi x)]$, and a Hookean elastic medium in which conservation of momentum is stated as
$$\rho\partial^2\boldsymbol{u}/\partial t^2=(\lambda+2\mu)\,\nabla\,(\nabla\cdot\boldsymbol{u})-\mu\nabla\times(\nabla\times\boldsymbol{u}) \tag{1}$$
where:
   - $\boldsymbol{u}(t,\boldsymbol{x})$ is the displacement [m]
   - ρ is the density [kg/m^3]
   - λ,μ are the bulk, shear elastic moduli [N/m^2].

   a) Is the longitudinal plane wave a solution of the equation of motion (1)?
   b) At what speed does the wave propagate?
   c) At an interface with another elastic medium (λ',μ') the incident wave $\boldsymbol{u}_i$ produces a reflected $\boldsymbol{u}_r$ and transmitted wave $\boldsymbol{u}_t$. Derive formulas for the reflection and transmission coefficients
$$R=\frac{\|\boldsymbol{u}_r\|}{\|\boldsymbol{u}_i\|},T=\frac{\|\boldsymbol{u}_t\|}{\|\boldsymbol{u}_i\|}$$

3. Consider a computational domain for (1) discretized by the second-order leap-frog scheme (LF), extending to distance L of each side of $x=0$. The subdomain $x<0$ is discretized with step size h, the subdomain $x>0$ is discretized with step size $2h$. At the left edge $x=-L$ an incoming plane wave $u_i(t,x)$ is entering the domain. The physics of the problem predicts unattenuated, non-dispersive passage of the wave, but the analysis within the Trefethen paper shows that the discrete scheme will induce both attenuation and dispersion.
   a) Compute the waveform at the interface $u_i(t,0_-)$ obtained after $u_i(t,-L)$ has propagated through the grid with step size h. Pay attention to the ξ/L ratio.
   b) Compute the reflection and transmission coefficients at the interface between the two grids
$$R=\frac{|u_r(t,0_-)|}{|u_i(t,0_-)|},T=\frac{|u_t(t,0_+)|}{|u_i(t,0_-)|}.$$
   c) Determine the ratio of change in material properties that would produce the same reflection and transmission coefficients for the exact wave equation as that produced by the grid spacing jump from h to $2h$ for the discretized wave equation.