1 Finite element model of plane wave scattering by sphere

Conservation of momentum for a Hookean elastic medium is described by

$$\rho \partial^2 \boldsymbol{u} / \partial t^2 = (\lambda + 2\mu) \nabla (\nabla \cdot \boldsymbol{u}) - \mu \nabla \times (\nabla \times \boldsymbol{u}). \tag{1}$$

The Helmholtz decomposition $\boldsymbol{u} = -\nabla \Psi + \nabla \times \boldsymbol{A}$ leads to wave equations for $\Psi, \boldsymbol{A} = (0, 0, A)$ are

$$\Psi_{tt} - c_p^2 \nabla^2 \Psi = 0, A_{tt} - c_s^2 \nabla^2 A = 0. \tag{2}$$

Looking for solutions of form $(\Psi, A) = (\psi, a) \exp(i\omega t)$ gives the Helmholtz equations

$$(\nabla^2 + k^2)\psi = 0, (\nabla^2 + l^2) a = 0, k = c_p / \omega, l = c_s / \omega. \tag{3}$$

Consider a solid body S sustaining both compressional and shear waves immersed in a fluid medium F with negligible shear modulus such that $a = 0$ in F. Let Ω denote the union of the fluid and solid domains, $\Omega = F + S$. On $\partial\Omega$ the displacement is therefore $\boldsymbol{u} = -\exp(i\omega t) \nabla \psi$, and the pressure on the fluid boundary is

$$p = \rho c_p \dot{\boldsymbol{u}} \cdot \boldsymbol{n} = -i\omega \rho c_p \exp(i\omega t) \frac{\mathrm{d}\psi}{\mathrm{d}n} = p_i + p_s, \tag{4}$$

with the incident-wave pressure is a plane wave

$$p_i(t, z) = -i p_0 \exp[i(\omega t - k z)]. \tag{5}$$

The far-field scattered pressure is given by the asymptotic expression (Farran, 1951)

$$p_s(t, r, \theta) = \frac{-i p_0}{k r} \exp(i\omega t) \sum_{n=0}^{\infty} a_n P_n(\cos\theta), \tag{6}$$

with θ the azimuthal angle in the spherical coordinate system $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$. The boundary value problem for ψ can be stated as

$$\begin{cases} (\nabla^2 + k^2)\psi = 0 & \text{in } \Omega \\ \dfrac{\mathrm{d}\psi}{\mathrm{d}n} = \dfrac{p_0}{\rho c_p^2} \dfrac{1}{k} \left[\cos(k r \cos\theta) + \dfrac{1}{k r} \displaystyle\sum_{n=0}^{\infty} a_n P_n(\cos\theta) \right] & \text{on } \partial\Omega \end{cases}, \tag{7}$$

with $k = \omega / c_p$ exhibiting a jump on ∂S due to change of material properties. The problem is axisymmetric with the gradient operator

$$\nabla = \frac{\partial}{\partial r} \boldsymbol{e}_r + \frac{\partial}{r \partial\theta} \boldsymbol{e}_\theta, \tag{8}$$

exhibiting a singularity at $r = 0$. In Cartesian coordinates the weak form obtained by multiplication with test function v and integration over $\Omega = [-a, a] \times [-a, a]$

$$\int_{-a}^{a} \int_{-a}^{a} \left[\frac{\partial\psi}{\partial x} \frac{\partial(v)}{\partial x} + \frac{\partial\psi}{\partial y} \frac{\partial v}{\partial y} - k^2 \psi v \right] \mathrm{d}x \, \mathrm{d}y - \oint_{\partial\Omega} v \frac{\mathrm{d}\psi}{\mathrm{d}n} \, \mathrm{d}l = 0.$$

The spherical coordinate weak form is obtained by multiplication with the test functions $r v$ (to avoid $r = 0$ singularity) and integration over Ω

$$\int_\Omega r v (\nabla^2 \psi + k^2 \psi) \, \mathrm{d}\Omega = 0. \tag{9}$$

Use of the Green formula

$$\int_\Omega (\varphi \nabla^2 \psi + \nabla\varphi \cdot \nabla\psi) \mathrm{d}\Omega = \oint_{\Sigma = \partial\Omega} \varphi (\nabla\psi \cdot \boldsymbol{n}) \, \mathrm{d}\Sigma, \tag{10}$$

gives

$$\int_\Omega r v (\nabla^2 \psi + k^2 \psi) \, \mathrm{d}\Omega = \int_\Omega (-\nabla(r v) \cdot \nabla\psi + k^2 r v \psi) \, \mathrm{d}\Omega + \oint_\Sigma r v \frac{\mathrm{d}\psi}{\mathrm{d}n} \, \mathrm{d}\Sigma = 0, \tag{11}$$

that is expressed in spherical coordinates as

$$\int_{-\pi}^{\pi}\int_{0}^{R}\left[r\,\frac{\partial\psi}{\partial r}\frac{\partial(r\,v)}{\partial r}+\frac{\partial\psi}{\partial\theta}\frac{\partial v}{\partial\theta}-k^2\,r^2\,\psi\,v\right]\mathrm{d}r\,\mathrm{d}\theta-R^2\int_{-\pi}^{\pi}v\,\frac{\mathrm{d}\psi}{\mathrm{d}r}\,\mathrm{d}\theta=0, \tag{12}$$

$$\int_{-\pi}^{\pi}\int_{0}^{R}\left[r\,\frac{\partial\psi}{\partial r}v+r^2\,\psi\,\frac{\partial v}{\partial r}+\frac{\partial\psi}{\partial\theta}\frac{\partial v}{\partial\theta}-k^2\,r^2\,\psi\,v\right]\mathrm{d}r\,\mathrm{d}\theta-R^2\int_{-\pi}^{\pi}v\,\frac{\mathrm{d}\psi}{\mathrm{d}r}\,\mathrm{d}\theta=0, \tag{13}$$

with $\psi(r,\theta)$ defined on the square domain $[0,R]\times[-\pi,\pi]$.