# MATH762 Homework 1 - Finite volume methods

Issued 1/31/12, Due 2/21/12

## BEARCLAW software

### Log on to a 64-bit Linux system

It is assumed work will be carried out on a 64-bit Linux system. Students have access to such a system by

```
ssh -X -l ONYEN pcoip01.its.unc.edu
```

from any other Linux system running an X-windows environment.

### Install BEARCLAW software

The following steps will download, configure and install the BEARCLAW finite volume PDE package.

```
cd ~

svn co http://mitran-lab.amath.unc.edu:8082/subversion/bearclaw

cp /usr/local/share/.bashrc .

cd bearclaw

cp /usr/local/share/Makefile.inc .

bash
```

### Test BEARCLAW software

Try to make and run a 2D advection example.

```
cd $BEARCLAW/examples/wavebear/2d/advection/example1

make dbear

dbear
```

The `make dbear` command makes the debug version of the code. You can use the Intel debugger invoked by `idb dbear` or the Totalview debugger invoked by `tv8 dbear` to debug the code.

After all errors have been eliminated faster execution is obtained by building the optimized version of the code:

```
make clean; make outclean; make distclean

make xbear
```

## Shallow water equations

### I. Solution using graphical aids

The shallow water system is

$$q_t + f(q)_x = 0$$

$$q = \begin{pmatrix} h \\ hu \end{pmatrix} = \begin{pmatrix} h \\ l \end{pmatrix}, f = \begin{pmatrix} uh \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} l \\ l^2/h + \frac{1}{2}gh^2 \end{pmatrix}$$

1. Compute the flux Jacobian $\partial f / \partial q$ eigensystem.

2. The Hugoniot loci are defined by

$$s\,(q_* - q) = f(q_*) - f(q)$$

Lay down a 11 x 11 grid of $q_* = (h_*, l_*)$ points in the state space $(h, l)$, $h = 0, 0.2, 0.4, \ldots, 2.0$, $l = -1, -0.8, \ldots, 1$. Use some graphics package (e.g. Mathematica) to plot the Hugoniot loci passing through each point $q_*$ of the grid. There are two such sets corresponding to shocks in each family of characteristics. Plot both.

3. The integral curves are defined by the ODEs

$$q'(\xi) = \frac{\mathrm{d}q}{\mathrm{d}\xi} = r(q(\xi)),$$

with $r$ an eigenvector of the flux Jacobian. Plot the integral curves passing through the points of the grid from Item 2 for both eigenvectors.

4. Consider the dam break problem

$$q(x, t = 0) = \begin{cases} (1.5, 0) & x < 0 \\ (0.5, 0) & x > 0 \end{cases}$$

Use the plots constructed in Items 2,3 to find $q(x, 1)$.

## II. Exact solution of the Riemann problem

1. Write the formulas describing Hugoniot loci and integral curves for the shallow water system

2. Write the nonlinear systems that describe intermediate states for the dam break problem from Item I.4.

3. Solve the nonlinear system to find an exact solution $q(x, 1)$. You can use any convenient nonlinear package (e.g. Mathematica)

## III. Finite volume solution using exact Riemann solver

1. Go to the BEARCLAW directory containing a template shallow water application for this homework assignment `cd $BEARCLAW/courses/MATH762_Spring2012/homework1/exactRS`

2. Follow the code comments in the `physflux` routine in `problem.f90` and implement an exact Riemann solver.

3. Execute the code and compare to results from Part II.

## IV. Roe linearization for shallow water equations

At an interface between two finite volume cells the Roe condition

$$\hat{A}_{i-1/2}(Q_i - Q_{i-1}) = f(Q_i) - f(Q_{i-1})$$

ensures exact representation of a jump in an eigendirection. The average flux Jacobian $\hat{A}_{i-1/2}$ can be computed as

$$\hat{A}_{i-1/2} = \int_0^1 \frac{\partial f}{\partial q}(q(\xi))\,\mathrm{d}\xi$$

along the path

$$q(\xi) = Q_{i-1} + (Q_i - Q_{i-1})\xi\,.$$

In a Roe linearization an auxiliary parameter vector is introduced by

$$z(\xi) = Z_{i-1} + (Z_i - Z_{i-1})\xi\,.$$

We have

$$f(Q_i) - f(Q_{i-1}) = \hat{C}_{i-1/2}(Z_i - Z_{i-1})$$

$$Q_i - Q_{i-1} = \hat{B}_{i-1/2}(Z_i - Z_{i-1})$$

$$\hat{C}_{i-1/2} = \int_0^1 \frac{\partial f}{\partial z}(z(\xi))\,\mathrm{d}\xi$$

$$\hat{B}_{i-1/2} = \int_0^1 \frac{\partial q}{\partial z}(z(\xi))\,\mathrm{d}\xi$$

$$\hat{A}_{i-1/2} = \hat{C}_{i-1/2}\hat{B}_{i-1/2}^{-1}$$

1. Carry out the computation above for

   $$z = \begin{pmatrix} \sqrt{h} \\ \sqrt{h u} \end{pmatrix}$$

   to obtain the Roe-average Jacobian

   $$\hat{A}_{i-1/2} = \begin{pmatrix} 0 & 1 \\ -\hat{u}^2 + g\bar{h} & 2\hat{u} \end{pmatrix}$$

   with

   $$\bar{h} = \frac{1}{2}(h_{i-1} + h_i), \hat{u} = \frac{\sqrt{h_{i-1}}\,u_{i-1} + \sqrt{h_i}\,u_i}{\sqrt{h_{i-1}} + \sqrt{h_i}}$$

2. Go to the template in `cd $BEARCLAW/courses/MATH762_Spring2012/homework1/Roe` and follow the comments to implement the Roe average solver.

3. Execute the code and compare with results from Parts II,III

### V. Arithmetic average

An alternative to the Roe-average solver is to naively set

$$\hat{A}_{i-1/2} = \frac{\partial f}{\partial q}\left(\frac{1}{2}(Q_{i-1} + Q_i)\right)$$

with $A_{i-1}, A_i$ the left and right flux Jacobians.

1. Go to the template in `cd $BEARCLAW/courses/MATH762_Spring2012/homework1/Arith` and follow the comments to implement the Roe average solver.

2. Execute the code and compare with results from Parts II,III,IV