Canonical hyperbolic PDE systems

Linear equations

Elasticity

Consider a one-dimensional bar subject to a stretching force at one end and with the other end fixed. The force produces at time $t>0$ a displacement $d(x,t)$ of an infinitesimal portion of the bar which was at position x in the initial state at $t=0$, before applying force $f(t)$. The speed at which the infinitesimal portion of the bar moves is

$$u(x,t)=\frac{\partial d(x,t)}{\partial t}=d_t(x,t)\,.$$

Since one end is fixed, the material's response to the applied force is the appearance of internal forces. The force per unit area of cross section is called the stress $\sigma(x,t)$. Adjacent elements might be deformed differently. The change in the x-direction of deformation is called the strain

$$\epsilon(x,t)=\frac{\partial d(x,t)}{\partial x}=d_x(x,t).$$

Note that

$$\epsilon_t-u_x=0\,.$$

Newton's law states that the force difference between two sides of an infinitesimal element produces a change in the momentum

$$\rho u_t-\sigma_x=0.$$

We have two equations for three unknown functions. The system is closed by some physical hypothesis on the response of the bar to imposed forces. The simplest hypothesis is the linear behavior given by Young's law

$$\sigma=E\epsilon,$$

with E known as Young's modulus or the tensile modulus. This allows forming a system

$$q_t+Aq_x=0$$

$$q=\left(\begin{array}{c}\sigma\\ u\end{array}\right), A=\left(\begin{array}{cc}0 & -E\\ -1/\rho & 0\end{array}\right).$$

The matrix A has the complete eigensystem

$$\lambda_1=-c, r_1=\left(\begin{array}{c}c\\ 1\end{array}\right), \lambda_2=c, r_2=\left(\begin{array}{c}-c\\ 1\end{array}\right),$$

with $c=\sqrt{E/\rho}$ the elastic wave speed through the material.

In matrix formulation

$$\Lambda=\left(\begin{array}{cc}-c & 0\\ 0 & c\end{array}\right), R=\left(\begin{array}{cc}c & -c\\ 1 & 1\end{array}\right), R^{-1}=\frac{1}{2}\left(\begin{array}{cc}1/c & -1/c\\ 1 & 1\end{array}\right)$$

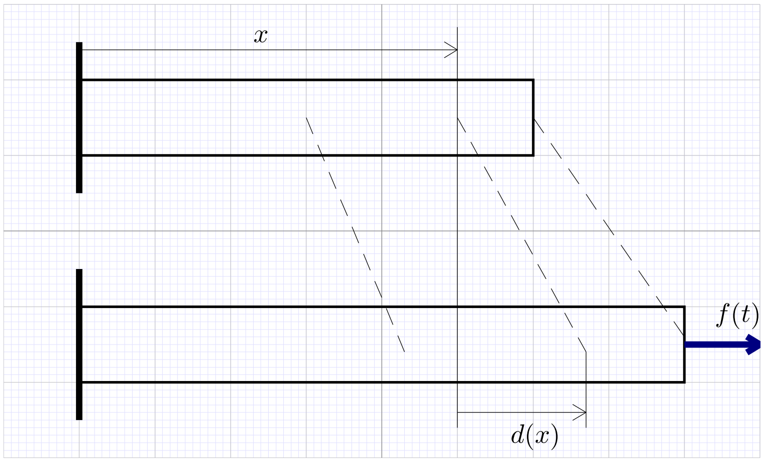

Figure 1. Deformation of a one-dimensional bar

Acoustics

Moving a piston in a cylinder containing a compressible gas will produce density $\rho(x,t)$ and pressure $p(x,t)$ increases which travel through the piston

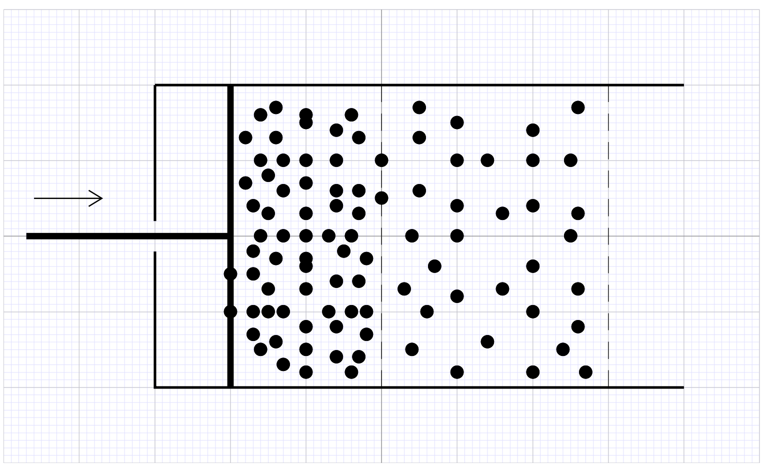

Figure 2. Moving a piston produces compression waves in a gas

The conservation of mass and momentum leads to equations

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u u + p)_x = 0$$

The density and pressure can be considered as composed of constant background values $(\rho_0, p_0, u_0 = 0)$ and perturbations (ρ', p', u'). If the perturbations are assumed to be small $\rho' \ll \rho_0$, $p' \ll p_0$, second-order effects (e.g. $\rho' u'$, $\rho_0 u' u'$) can be neglected leading to

$$\rho'_t + \rho_0 \, u'_x = 0,$$

$$\rho_0 u'_t + p'_x = 0.$$

Again, a constitutive equation is needed to close the system. For acoustics the compression is assumed to occur sufficiently fast that no heat conduction takes place, the process is said to isentropic. In such cases we pressure perturbations are linked to density perturbations by the isentropic transformation law

$$\frac{p(\rho)}{p_0} = \frac{\rho^\gamma}{\rho_0^\gamma}$$

with γ the adiabatic coefficient. Take logarithm

$$\ln p - \ln p_0 = \gamma(\ln \rho - \ln \rho_0)$$

and differentiate to obtain

$$\frac{\mathrm{d}p}{p} = \gamma \frac{\mathrm{d}\rho}{\rho},$$

For small perturbations we therefore have

$$\frac{p'}{p_0} = \gamma \frac{\rho'}{\rho_0} \Rightarrow p' = c^2 \rho'$$

with

$$c = \sqrt{\gamma \frac{p_0}{\rho_0}}$$

the sound speed in the medium. The system

$$q_t + A q_x = 0$$

$$q = \begin{pmatrix} p \\ u \end{pmatrix}, A = \begin{pmatrix} 0 & \rho_0 c^2 \\ 1/\rho_0 & 0 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix}, R = \begin{pmatrix} -\rho_0 c & \rho_0 c \\ 1 & 1 \end{pmatrix}, R^{-1} = \frac{1}{2} \begin{pmatrix} -1/\rho_0 c & 1/\rho_0 c \\ 1 & 1 \end{pmatrix}$$

Non-linear equations

Shallow water equations

Consider the flow of water of varying depth $h(x, t)$ over distances much larger than h. Variations in the vertical direction can be neglected leading to the system

$$q_t + f(q)_x = 0$$

$$q = \begin{pmatrix} h \\ hu \end{pmatrix}, f = \begin{pmatrix} uh \\ hu^2 + \frac{1}{2} g h^2 \end{pmatrix}$$

Euler equations for gas dynamics