CHAPTER 2

# Computational Fluid Dynamics

Historically, computational fluid dynamics (CFD) has been one of the first disciplines in which numerical methods have been applied widely. The main reason underlying the early adoption of computational methods is the nonlinearity of the equations of motion which leads to significant analytical difficulties. Indeed, even now after more than a century of research we do not know whether a solution to the general problem of fluid motion exists in a mathematical sense. From the physics point of view the question of existence of solutions does not arise - experiment ensures us that fluid motions do exist for appreciable intervals of time.

## 1. On the definition of a "fluid"

Fluids are understood from common experience to describe a state of matter in which a substance assumes the shape of the walls bounding it by contrast to the solid state. Liquids may fill only a certain part of the domain open to it while gases will fill the container in its entirety. At the microscopic level the essential characteristic of a fluid is that the interactions between particles (atoms, molecules) are weak. For gases they range from interactinos so weak that they only occur when the particles "collide" to slightly longer range interactions. By collision we typically understand that the particles approach distances comparable to the diameter of the electron cloud surrounding an atom in the composition of the gas. For all gases two-particle interactions are the dominant contribution to the overall behavior of the gas. Liquids exhibit stronger interaction forces such that the motion of instantaneous clusters of particles becomes highly correlated over small intervals of time. In contrast, crystalline solids have strong particle interactions leading to an overall collective motion observed macroscopically.

By extension of the common concept of a fluid, any system for which interactions between the component particles are weak and which exhibits a large number of component parts can be treated as a fluid. In cosmology galaxies are small compared to the scale of the Universe and interactions between them are weak - gravitational forces play an important role only when galaxies are very close together in a state of "collision". Hence many cosmologial simulations are carried out using the equations of fluid mechanics. The same equations are used in the analysis of traffic on highways (though one hopes to avoid collisions in this context). The spread of a malignant tumour can be modeled using the equations of fluid dynamics. We can see that though initially motivated by problems in hydro- and aerodynamics, the equations and the methods considered in this chapter have wide-ranging applicability throughout applied mathematics.

Since the component parts of a fluid are too many and too small (we say that they are below the scale of resolution we are interested in) we must introduce quantities that generalize the common concepts of point mass, point velocity. This is

done by introducing density functions for the main conserved quantities in mechanics: mass, momentum and energy. The analogy is set forth in the following table. Here $V$ is a volume containing a fluid and $\sigma(V)$ is the measure of the volume $V$. Often, we are not so precise and use $V$ both for the volume and its measure. It is common terminology to refer to the "density" of a fluid when referring to the density of mass. Similarily it is common practice to speak of the momentum of a fluid $\rho\vec{u}$ though this is more properly referred to as the density of momentum. The macroscopic definition of these quantities involves taking a limit $\sigma(V) \rightsquigarrow 0$ in which the measure of the volume becomes very small. Mathematically it does not go to zero but rather to a value such that a typical distance $d = [\sigma(V)]^{1/3}$ is so small that the individual component particles of the fluid become distinguishable. It becomes apparent that some sort of separation of scales is involved: we are interested in establishing the motion of the particles of fluid on scales much large than the distance of interaction between the particles themselves. This is formalized in the *Knudsen* number

$$Kn = \frac{\lambda}{l} \tag{1.1}$$

where $\lambda$ is a quantity indicative of the microscopic interactions between the particles and $l$ is the scale of motion we are interested in describing. If the Knudsen number is small, e.g. $Kn < 10^{-3}$ the system resembles a fluid. If it has intermediate values $10^{-3} < Kn < 10^{-1}$ a kinetic description is used and if it is large the system is described using point mechanics. For a gas $\lambda$ is typically the mean free path, i.e. the distance a particle traverses before it, on average, collides with another particle. A typical practical situation might be the computation of air flow around an airplane so $l = 10^{-2}$ m is a reasonable estimate of the scales of motion we would be interested in. By comparison, the mean free path for air is on the order $\lambda = 10^{-7}$ m so $Kn = 10^{-5}$.

The table also shows how we can define fluid density quantities microscopically. The density of mass may be defined by summing the masses $m_i$ of all the particles having a position vector $\vec{x}_i$ that places them within the volume $V$ and dividing by the measure of the volume $\sigma(V)$.

| Point particle quantity | Analagous fluid quantity | Macroscopic definition | Microscopic definition |
|---|---|---|---|
| mass - $m$ | density (of mass) $\rho$ | $\rho = \lim_{\sigma(V)\rightsquigarrow 0} \frac{m(V)}{\sigma(V)}$ | $\rho = \frac{1}{\sigma(V)} \sum_{\vec{x}_i \in V} m_i$ |
| momentum - $m\vec{u}$ | (density of) momentum $\rho\vec{u}$ | $\rho = \lim_{\sigma(V)\rightsquigarrow 0} \frac{m(V)\vec{u}(V)}{\sigma(V)}$ | $\rho = \frac{1}{\sigma(V)} \sum_{\vec{x}_i \in V} m_i\vec{u}_i$ |
| energy - $E$ | (density of) energy $\rho E$ | $\rho = \lim_{\sigma(V)\rightsquigarrow 0} \frac{m(V)E(V)}{\sigma(V)}$ | $\rho = \frac{1}{\sigma(V)} \sum_{\vec{x}_i \in V} m_i E_i$ |

## 2. The conservation equations

When the velocity of the fluid is small by comparison to the speed of light we say that we have classical fluid motion. The equations of motion are derived from the general physical principle of conservation of mass, momentum, energy. Recall that the local, differential form of a conservation principle is

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{f} = \sigma \tag{2.1}$$

with $q$ - the conserved quantity, $\mathbf{f}$ - the flux of $q$ and $\sigma$ - the source of $q$ (from hereon we shall use $\sigma$ as a notation for source intensities and adopt the common practice in physics of using $V$ for both volume and its measure). It is a physicist's job to determine the appropriate expressions for $q, \mathbf{f}, \sigma$. One the most lucid presentations by physicists of how this is done can be found in the classic *Fluid Mechanics* by Landau and Lifschitz. We'll go over the main points in the derivation.

**2.1. Conservation of mass - the continuity equation.** To express the conservation of mass we set $q = \rho$ and consider now the means by which mass may be transferred into a control volume $V$. One way is by overall, macroscopic motion of the fluid. Another way is through microscopic diffusion processes for multi-species fluids (fluids containing more than one chemical component). Let us concentrate only on single-species fluids for now. It is clear that the macroscopic flux of mass (mass transported per unit time thorugh unit area) is just the density multiplied by the velocity

$$\vec{f} = \rho\vec{u} \ . \tag{2.2}$$

If there are no chemical reactions then mass is neither created or destroyed and we come across the familiar continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{u}) = 0 \ , \tag{2.3}$$

or

$$\rho_{,t} + (\rho u_i)_{,i} = 0 \ , \tag{2.4}$$

in component form. An important special case of the continuity equation is obtained when we consider a fluid to be incompressible. This is satisfied to a high degree of accuracy for common fluids such as liquid water for instance. Then we have $\rho = const$ and the continuity equation becomes

$$\nabla \cdot \vec{u} = 0 \tag{2.5}$$

or

$$u_{i,i} = 0 \ . \tag{2.6}$$

**2.2. Conservation of momentum - the Navier-Stokes equations.** Now let us turn to the more complicated situation of momentum transport. We have $\vec{q} = \rho\vec{u}$ or $q_i = \rho u_i$ componentwise. Again momentum can be transported through a control volume either by macroscopic motion of the fluid, a process called *convection*, or through microscopic processes termed *diffusion*. The macroscopic contribution to the flux is straightforward

$$\mathbf{f}^c = \rho\vec{u} \otimes \vec{u} \ . \tag{2.7}$$

In component form we have

$$f^c_{ij} = \rho u_i u_j \tag{2.8}$$

At the microscopic level there are now processes which have to be taken into account even for a single-species fluid. As particles pass through any bounding surface of a control volume they carry along a certain momentum. The momentum of an individual particle is not necessarily that of the overall fluid at that point - a mismatch arises and this is felt macroscopically as a "pressure" or a "tangential stress". Pressure is essentially the microscopic transport of the momentum oriented

normal to a surface. Momentum is a vector quantity and tangential momentum is also transferred to a control volume when a particle crosses a volume surface. This is felt macroscopically as "friction" or a "tangential stress". Both of these correspond to the same physical process: transport of momentum at the microscopic scales. The different labels are more a result of historical accident than any true difference between pressure and friction.

Empirical observations suggest that pressure is an intrinsic isotropic scalar and friction is proportional to the gradient of the macroscopic fluid velocity. These physical quantities should be invariant under translations and rotations, and tensor analysis gives the only possible expression for such a description of the microscopic flux of momentum as

$$f^d_{ij} = p\delta_{ij} - a(u_{i,j} + u_{j,i}) - bu_{l,l}\delta_{ij} \ . \tag{2.9}$$

It is convenient to rewrite this expression as

$$f^d_{ij} = p\delta_{ij} - \eta\left(u_{i,j} + u_{j,i} - \frac{2}{3}u_{l,l}\delta_{ij}\right) - \zeta u_{l,l}\delta_{ij} \ , \tag{2.10}$$

which is just a different way of expressing the scalar constants $a, b$. The advantage of this second form is that the tensor multiplying $\eta$ has a zero trace. Recall that the trace of a tensor $A_{ij}$ is a scalar $T = A_{ij}\delta_{ij}$ (double summation on the repeated indices $i, j$) so

$$\left(u_{i,j} + u_{j,i} - \frac{2}{3}u_{l,l}\delta_{ij}\right)\delta_{ij} = u_{i,j}\delta_{ij} + u_{j,i}\delta_{ij} - \frac{2}{3}u_{l,l}\delta_{ij}\delta_{ij} \tag{2.11}$$

$$= u_{i,i} + u_{i,i} - \frac{2}{3}u_{l,l}(3) = 0 \ . \tag{2.12}$$

The two parameters $\eta$, $\zeta$ are called the first and second coefficients of viscosity, respectively.

This leads to an overall flux $\mathbf{f} = \mathbf{f}^c + \mathbf{f}^d$ which is a two-component tensor (i.e. a matrix) and whose components are

$$f_{ij} = \rho u_i u_j + p\delta_{ij} \ - \eta\left(u_{i,j} + u_{j,i} - \frac{2}{3}u_{l,l}\delta_{ij}\right) - \zeta u_{l,l}\delta_{ij} \ . \tag{2.13}$$

The final element needed to write down the conservation of momentum is an expression for the source term $\sigma$. Dynamics teaches us that forces are the sources of momentum. Let us suppose that the volume-distributed force within the fluid is given by $\sigma_i = \rho g_i$.

The local, differential form of the conservation of momentum can now be written in vector form as

$$\frac{\partial(\rho\vec{u})}{\partial t} + \nabla\cdot\mathbf{f} = \rho\vec{g} \ . \tag{2.14}$$

Using the compact component notation, the conservation of momentum is

$$(\rho u_i)_{,t} = \rho g_i - \left[\rho u_i u_j + p\delta_{ij} \ - \eta\left(u_{i,j} + u_{j,i} - \tfrac{2}{3}u_{l,l}\delta_{ij}\right) - \zeta u_{l,l}\delta_{ij}\right]_{\ ,j} \tag{2.15}$$

$$= \rho g_i - \left(\rho u_i u_j\right)_{,j} - p_{,j}\delta_{ij} + \left[\eta\left(u_{i,j} + u_{j,i} - \tfrac{2}{3}u_{l,l}\delta_{ij}\right)\right]_{\ ,j} + \left[\zeta u_{l,l}\delta_{ij}\right]_{\ ,j} \tag{2.16}$$

$$= \rho g_i - \left(\rho u_i u_j\right)_{,j} - p_{,i} + \left[\eta\left(u_{i,j} + u_{j,i} - \tfrac{2}{3}u_{l,l}\delta_{ij}\right)\right]_{\ ,j} + \left[\zeta u_{l,l}\right]_{\ ,i} \tag{2.17}$$

Note that up to this point we have not given any specific physical significance to the quantities $p$, $\eta$, $\zeta$. They have been introduced as scaling coefficients in

the general expression of a flux independent of translations, rotations and that depends linearly upon velocity gradients. We must assign physical significance to these parameters so that experiments can give values for these parameters. We must also consider what kind of boundary conditions are to be imposed on the the unknown quantities.

The microscopic momentum flux density is also the stress felt by an infinitesimal fluid element

$$S_{ij} = f^d_{ij} = p\delta_{ij} - \eta\left(u_{i,j} + u_{j,i} - \frac{2}{3}u_{l,l}\delta_{ij}\right) - \zeta u_{l,l}\delta_{ij}. \tag{2.18}$$

The trace of the stress is

$$S_{ij}\delta_{ij} = 3(p - \zeta u_{l,l}). \tag{2.19}$$

In a fluid at rest we typically assign the label "hydrostatic pressure" $P$ to the average normal stress $\frac{1}{3}(S_{11} + S_{22} + S_{33})$. We would like to be able to extend this label to arbitrary fluid motions, i.e. to have $P = p$. This is possible if either we have:

(1) $u_{l,l} = 0$, i.e. the fluid is incompressible, or
(2) $\zeta = 0$ the second coefficient of viscosity is zero. This is known as the Stokes hypothesis and is verified for a wide variety of fluid motions (a notable exception is ultrasonic vibrations in a fluid).

Assuming now that we can indeed identify the scalar $p$ with the hydrostatic pressure $P$, the equations of motion take the form

$$(\rho u_i)_{,t} = \rho g_i - (\rho u_i u_j)_{,j} - p_{,i} + \left[\eta\left(u_{i,j} + u_{j,i} - \tfrac{2}{3}u_{l,l}\delta_{ij}\right)\right]_{,j} \tag{2.20}$$

and are known as the *Navier-Stokes equations*. There remains only one more parameter to discuss, $\eta$. A simple flow which exhibits fluid friction is to enclose a fluid between two plates, keep the plate at $x_2 = 0$ stationary and move the plate at $x_2 = h$ with a fixed velocity $\vec{U} = U\,\vec{e}_1$. This is known as a *Couette* flow and the solution of the equations of motion for such a flow is a linear distribution of the velocity in the direction of the plate motion

$$u_1 = Ax_2 = \frac{\partial u_1}{\partial x_2}x_2. \tag{2.21}$$

The constant velocity gradient can be determined from assuming that the fluid velocity at a solid-wall boundary is equal to the velocity of the wall. These are known as *no-slip boundary conditions*. We have

$$\frac{\partial u_1}{\partial x_2} = \frac{U - 0}{h} = U/h \tag{2.22}$$

One can measure the force needed to keep one of the plates in motion and this, in turn is an indication of the fluid friction. Experiments show that the friction increases linearly with the top plate velocity $U$ and the inverse of the plate spacing. This led Newton to propose a definition of the tangential stress exerted between two fluid elements as

$$\tau = \eta\frac{\partial u_1}{\partial x_2} \tag{2.23}$$

thus assigning a specific physical significance to the parameter $\eta$.

**2.3. Conservation of energy.** The derivation of an equation expressing the conservation of energy is a bit more involved because there are multiple definitions of what we mean by "energy" in the context of fluid flow. For a point mass one typically starts out with "energy" signifying the kinetic energy $K = \frac{1}{2}mv^2$. Afterwards, when considering motion in a gravitational field, it makes sense to assign to the point mass a "total energy" $E = \frac{1}{2}mv^2 + mgh$ with $mgh$ called the potential energy of the point mass in a gravitational field. These are not the only definitions of "energy" we could use; the point mass is a model of a true solid that is composed of atoms that move and hence have an intrinsic kinetic energy. This type of energy is typically not included in point dynamics. For fluids a similar problem arises in that we can assign various definitions of what we mean by energy and write down corresponding transport equations. Here, we'll consider just the simplest possible definition and look at the change in time of kinetic energy for an incompressible fluid with constant $\eta$ and with no external forces $\vec{g} = 0$. We'll adopt a slightly different approach to determining the local form of the conservation form. Instead of first determining an expression for the flux, we'll directly compute the derivative with respect to time of the kinetic energy (density)

$$\frac{\partial}{\partial t}K = \frac{\partial}{\partial t}\frac{\rho u^2}{2} = \rho u_i \frac{\partial u_i}{\partial t} = \rho u_i u_{i,t} \ . \tag{2.24}$$

The Navier-Stokes equations give us an expression for

$$(\rho u_i)_{,t} = \rho u_{i,t} = -\left(\rho u_i u_j\right)_{,j} - p_{,i} + \left[\eta\left(u_{i,j} + u_{j,i} - \tfrac{2}{3}u_{l,l}\delta_{ij}\right)\right]_{,j} \tag{2.25}$$

$$= -\rho u_{i\,,j} u_j - \rho u_i u_{j,j} - p_{,i} + \left[\eta\left(u_{i,j} + u_{j,i} - \tfrac{2}{3}u_{l,l}\delta_{ij}\right)\right]_{,j}. \tag{2.26}$$

Note that for an incompressible fluid we have $u_{j,j} = 0$ so

$$(\rho u_i)_{,t} = -\rho u_{i\,,j} u_j - p_{,i} + \eta\left(u_{i,j} + u_{j,i}\right)_{,j} \tag{2.27}$$

Therefore

$$\frac{\partial}{\partial t}\frac{\rho u^2}{2} = \rho u_i u_{i,t} = -\rho u_i u_{i\,,j} u_j - u_i p_{,i} + \eta u_i \left(u_{i,j} + u_{j,i}\right)_{,j} . \tag{2.28}$$

We'll now try to use this expression to come up with what the flux of kinetic energy is. For this we must isolate the divergence of vector quantity in the above equation. Note that

$$(pu_i)_{,i} = p_{,i}u_i + pu_{i,i} = u_i p_{,i} \tag{2.29}$$

for an incompressible fluid. Similarily

$$\left(\tfrac{1}{2}\rho u_i u_j u_j\right)_{,i} = \tfrac{1}{2}\rho u_{i,i} u_j u_j + \tfrac{1}{2}\rho u_i u_{j,i} u_j + \tfrac{1}{2}\rho u_i u_j u_{j,i} = \rho u_i u_j u_{j,i} = \rho u_j u_i u_{i\,,j} \ . \tag{2.30}$$

The last equality above corresponds to switching the $i, j$ indices. The viscous term from (2.28) can be rewritten as

$$\eta u_i \left(u_{i,j} + u_{j,i}\right)_{,j} = \eta u_j \left(u_{j,i} + u_{i,j}\right)_{,i} \tag{2.31}$$

by switching indices and then as

$$\eta u_j \left(u_{j,i} + u_{i,j}\right)_{,i} = \left[\eta u_j \left(u_{j,i} + u_{i,j}\right)\right]_{,i} - \eta u_{j,i}\left(u_{j,i} + u_{i,j}\right) \tag{2.32}$$

We can rewrite (2.28) as

$$\frac{\partial}{\partial t}\frac{\rho u^2}{2} = \rho u_i u_{i,t} = -\left[\tfrac{1}{2}\rho u_i u_j u_j + p u_i - u_j \eta\left(u_{i,j} + u_{j,i}\right)\right]_{,i} - \eta u_{j,i}\left(u_{j,i} + u_{i,j}\right) \ , \tag{2.33}$$

and can now identify the flux and source terms for the conservation of kinetic energy. The flux has components

$$f_i = \tfrac{1}{2}\rho u_i u_j u_j + p u_i - \eta\left(u_{i,j} + u_{j,i}\right) u_j \tag{2.34}$$

which in vector form corresponds to

$$\vec{f} = K\vec{u} + p\vec{u} - \mathbf{S}\cdot\vec{u} \tag{2.35}$$

with $\mathbf{S}$ the viscous stress tensor

$$S_{ij} = \eta\left(u_{i,j} + u_{j,i}\right) \ . \tag{2.36}$$

The physical effects present in the flux are:

(1) $K\vec{u}$ - convective (macroscopic) transport of kinetic energy by the fluid
(2) $p\vec{u}$ - microscopic transport of energy thorugh the action of normal stresses in the fluid. This is more commonly interpreted macroscopically as work done by the pressure forces
(3) $-\mathbf{S}\cdot\vec{u}$ - microscopic transport of energy thorugh the action of tangential stresses in the fluid. This is more commonly interpreted as work consumed to overcome friction forces.

The source term is

$$\sigma = -\eta u_{j,i}\left(u_{j,i} + u_{i,j}\right) \ . \tag{2.37}$$

This corresponds to loss of kinetic energy (into thermal energy) due to the action of friction forces.

**2.4. Boundary conditions.** The theory of boundary conditions for the fluid dynamic equations can get quite involved[1]. Rather than going into the details of what boundary conditions lead to well-posed problems in a precise mathematical sense we'll adopt the viewpoint of physics which observes that fluid motions exist and are unique for specific initial and boundary conditions.

2.4.1. *Solid walls.* At a solid wall it is common experience that fluids "stick" to the wall. This leads to the so-called *no-slip boundary condition* where the fluid velocity at a solid wall $\vec{u}_w$ is the same as the velocity of the wall itself $\vec{V}$

$$\vec{u}_w = \vec{V} \ . \tag{2.38}$$

There are several departures from this boundary condition that are encountered experimentally. Some fluids like liquid helium exhibit what is known as superfluidity and fluid velocity might be different from the wall velocity. Also in the transition region between fluid and particle behavior when Knudsen numbers become large ($Kn \gtrsim 0.01$) slip is observed between the fluid velocity and the wall velocity – such situations arise in space vehicle atmosphere re-entry. We shall not discuss these rather specific situations and assume that the no-slip boundary condition is valid for viscous fluids. Note that the no-slip boundary condition only gives us 3 relations for the 5 unknowns needed to describe fluid motion. Further boundary conditions must be imposed. These typically have to do with the specifics of heat transfer of the wall. A general discussion may be found in fluid dynamics texts. Generally we'll assume we have *adiabatic walls*, i.e. there is no heat transferred from the wall to the fluid and work out the implications of this assumption as the need arises.

[1]See O.A. Ladyzhenskaya, *Boundary value problems of mathematical physics* for a mathematical treatment of these issues.

2.4.2. *Free surfaces, fluid-fluid interfaces.* Free surfaces of a fluid arise in many situations: the ocean surface, two immiscible liquids in the same container are some some examples. Though one might assume at first sight that the required boundary condition is that fluids should move with the same velocity this is not a necessary condition that comes from the fluid dynamic equations. If one isolates a fluid-fluid interface with a small control volume and lets the thickness of this control volume go to zero the forces acting on the control volume sides must balance, otherwise the fluid inside would be accelerated to an arbitrarily large value. The proper boundary conditions to apply in such situations are known as stress continuity equations and state that on both sides of an interface the motion of the fluids must be such that the normal and tangential stresses match across the interface.

## 3. Computational techniques for specific fluid models

Though we could attack the problem of directly solving the full set of conservation laws governing the motion of a fluid it is more instructive to first consider specific simplified situations and the techniques which we can use for these cases.

**3.1. Inviscid fluids.** An inviscid fluid has negligible viscosity $\eta \cong 0$ in most of the flow domain of interest. Many applications can be treated under this hypothesis, e.g. aerodynamics of aircraft, hydrodynamics of ships, in order to obtain a first estimate of the forces exerted by a fluid on other objects. For most of these applications thermodynamics provides additional informatino which can be used to eliminate the need for solving the energy equation explicitly. We are left with a system formed by the continuity and momentum equations which reads

$$\rho_{,t} + (\rho u_j)_{,j} = 0 \tag{3.1}$$

$$(\rho u_i)_{,t} + (\rho u_i u_j)_{,j} = \rho g_i - p_{,i} \tag{3.2}$$

The momentum equation can be rewritten as

$$\rho_{,t} u_i + \rho u_{i,t} + u_i(\rho u_j)_{,j} + \rho u_j u_{i,j} = \rho(u_{i,t} + u_j u_{i,j}) + u_i\left[\rho_{,t} + (\rho u_j)_{,j}\right] = \rho g_i - p_{,i}. \tag{3.3}$$

Using the continuity equation this becomes

$$u_{i,t} + u_j u_{i,j} = \rho g_i - \frac{1}{\rho} p_{,i} \tag{3.4}$$

and is known as the *Euler equation* of fluid dynamics.

In vector form we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \tag{3.5}$$

$$\frac{D\vec{u}}{Dt} \equiv \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = \vec{g} - \frac{\nabla p}{\rho} \tag{3.6}$$

Note the appearance of the substantive derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$ expressing the change in a qunatity due to the combine effects of its rate of change in time and the difference between inflow and outflow of that quantity. From the vector identity $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ one can obtain a very useful expression of the substantive derivative. Consider the product $\vec{u} \times (\nabla \times \vec{u})$. Remembering that the curl should only act upon the second instance of $\vec{u}$ we can rewrite this product as

$$\vec{u} \times (\nabla \times \vec{u}) = \nabla \left( \frac{u^2}{2} \right) - (\vec{u} \cdot \nabla)\, \vec{u}\ , \tag{3.7}$$

and therefore obtain an expression for the substantive derivative

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{u^2}{2}\right) + (\nabla \times \vec{u}) \times \vec{u}. \tag{3.8}$$

The curl of the vector field $\vec{u}$ which describes the local, instantaneous rate of rotation of the fluid and is known as the vorticity $\vec{\omega}$

$$\vec{\omega} = \nabla \times \vec{u}\ , \tag{3.9}$$

so the Euler equations can also be written as

$$\frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{u^2}{2}\right) + \vec{\omega} \times \vec{u} = \vec{g} - \frac{\nabla p}{\rho}\ . \tag{3.10}$$

Taking the curl of this equation leads to the *vorticity transport equation* also known as the *Helmholtz equation*

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u}\cdot\nabla)\vec{\omega} = (\vec{\omega}\cdot\nabla)\vec{u} - (\nabla\cdot\vec{u})\vec{\omega} + \nabla\times\vec{g} - \nabla\times\left(\frac{\nabla p}{\rho}\right)\ . \tag{3.11}$$

3.1.1. *The Bernoulli relation.* Notice that up to this point we have used the conservation of mass and conservation of momentum principles. In order to close the system of equations decribing inviscid fluid motion we need to also invoke conservation of energy. There are various ways to do this. We shall see later on that one possibility is to impose a specific form of the thermodynamic process which the fluid undergoes during its motion. When imposing such a relation the first principle of thermodynamics (an affirmation of the conservation of energy) is used.

Another important way in which conservation of energy can be used is actually derived from the Euler equations. Assume that we have a *stationary motion*, i.e. there is no change of the flow variables with respect to time. The Euler equations then read

$$\nabla\left(\frac{u^2}{2}\right) + \vec{\omega} \times \vec{u} = \vec{g} - \frac{\nabla p}{\rho}\ . \tag{3.12}$$

It is typically the case that the external forces are conservative and hence can be expressed as the gradient of a potential

$$\vec{g} = \nabla U\ . \tag{3.13}$$

Furthermore for almost all fluids one can express the ratio $\nabla p/\rho$ as the gradient of what is known as a *barotropic potential* function $\mathcal{P}$

$$\frac{\nabla p}{\rho} = \nabla\mathcal{P} \tag{3.14}$$

so the steady Euler equations are

$$\nabla\left(\frac{u^2}{2}\right) + \vec{\omega} \times \vec{u} = \nabla\left(U - \mathcal{P}\right) \tag{3.15}$$

Multiply this by some arbitrary displacement $d\vec{r}$ and let the $d$-operator denote changes in the flow variables made along this displacement

$$d\left(\right) + (\vec{\omega} \times \vec{u}) \cdot d\vec{r} = d\left(U - \mathcal{P}\right)\ . \tag{3.16}$$

The mixed product above is null for two cases

(1) If $d\vec{r}$ corresponds to a streamline, i.e. it is colinear with $\vec{u}$;
(2) If $d\vec{r}$ corresponds to a vorticity line, i.e. it is colinear with $\vec{\omega}$.

In both of the above cases we obtain

$$d\left(\frac{u^2}{2} - U + \mathcal{P}\right) = 0 \tag{3.17}$$

which is the general form of the *Bernoulli equation*. This equation is essentially an expression of the conservation of energy and provides the final relation necessary for solving the system of PDE's describing the motion of an inviscid fluid. You may be most familiar with the form of the Bernoulli equation for incompressible fluids in a gravitation field for which $U = -gz$ and $\mathcal{P} = p/\rho$ with $z$ the height above some reference line

$$\frac{u^2}{2} + gz + \frac{p}{\rho} = const\ . \tag{3.18}$$

3.1.2. *The Crocco relation.* Though we will not go into the detailed thermodynamics of fluid flow there is one relation that is very useful that should be mentioned. It provides a link between kinematic and thermodynamic parameters and is known as the Crocco relation. We start from the first principle of thermodynamics in differential form

$$de = dq - dl \tag{3.19}$$

with $e$ - the internal energy, $q$ - heat transfer from the fluid to surrounding systems and $l$ - the mechanical work exchanged between the fluid and the surroundings. The definition of entropy $s$ is given differentially by

$$ds = \frac{dq}{T} \tag{3.20}$$

with $T$ the temperature. Mechanical work is given by

$$dl = pd\left(\frac{1}{\rho}\right) \tag{3.21}$$

so we have

$$de = Tds - pd\left(\frac{1}{\rho}\right) \tag{3.22}$$

from where

$$\nabla e = T\ \nabla s - p\nabla\left(\frac{1}{\rho}\right)\ . \tag{3.23}$$

In the Euler equations we have the term $\nabla p/\rho$ which can be rewritten

$$\frac{\nabla p}{\rho} = \nabla\left(\frac{p}{\rho}\right) - p\nabla\left(\frac{1}{\rho}\right) = \nabla\left(e + \frac{p}{\rho}\right) - T\nabla s \tag{3.24}$$

so the Euler equations become

$$\frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{u^2}{2}\right) + \vec{\omega} \times \vec{u} = -\nabla\left(e + \frac{p}{\rho}\right) + T\nabla s \tag{3.25}$$

or

$$T\nabla s = \frac{\partial \vec{u}}{\partial t} + \nabla\left(e + \frac{p}{\rho} + \frac{u^2}{2}\right) + \vec{\omega} \times \vec{u}\ , \tag{3.26}$$

the *Crocco relation*. The quantity $h = e + p/\rho$ is known as the enthalpy of a fluid and expresses its internal energy including the work necessary to form the fluid from its constituent particles. Energy in a fluid may be transferred between kinetic

and internal forms. In an inviscid fluid this process is lossless and it is convenient to have a measure of the overall constant total energy available to the fluid. For this we define the *stagnation energy*

$$E = e + \frac{u^2}{2} \ , \tag{3.27}$$

and the *stagnation enthalpy*

$$H = h + \frac{u^2}{2} \ . \tag{3.28}$$

Using these quantities Crocco's relation can be written as

$$T\nabla s = \frac{\partial \vec{u}}{\partial t} + \nabla H + \vec{\omega} \times \vec{u} \ . \tag{3.29}$$

Notice that for a steady motion ($\partial\vec{u}/\partial t = 0$) of a lossless fluid ($\nabla H = 0$) the change in entropy is given only by the vector product of vorticity and velocity

$$T\nabla s = \vec{\omega} \times \vec{u} \ . \tag{3.30}$$

3.1.3. *Inviscid, incompressible fluids - Potential flow.* If the fluid is both incompressible $\rho = const$, and inviscid $\eta = 0$ the equations of motion simplify considerably. The continuity equation becomes

$$u_{i,i} = 0 \ . \tag{3.31}$$

The momentum equations are

$$u_{i,t} + u_j u_{i,j} = -p_{,i} \tag{3.32}$$

in the absence of external forces. In vector notation form the equations are

$$\nabla \cdot \vec{u} = 0 \tag{3.33}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p \tag{3.34}$$

with the density assumed to be equal to 1 (always possible through a change of units of measurement). Taking the curl of the incompressible Euler equations leads to the following special form of the vorticity transport equation

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla)\vec{\omega} = (\vec{\omega} \cdot \nabla)\vec{u} \ . \tag{3.35}$$

This shows that if the initial vorticity of an inviscid fluid is zero it remains zero at all later times within the interior of the fluid domain

$$\vec{\omega}_0 = 0 \Rightarrow \frac{\partial \vec{\omega}}{\partial t} = 0 \ . \tag{3.36}$$

The velocity field is said to be *irrotational* or *solenoidal* and can be expressed as the gradient of a scalar function known as the velocity potential $\phi$

$$\vec{u} = \nabla\phi \ . \tag{3.37}$$

This leads to a great simplification of the system of PDE's describing fluid motion since we only seek a single scalar function and furthermore we see that this function is harmonic

$$\nabla^2\phi = 0 \tag{3.38}$$

as a result of the continuity equation. Since the fluid velocity is determined from a single scalar potential such flows are called *potential flows* and we can apply the

full range of methods known to be applicable to the Laplace equation to determine the fluid motion. From the Crocco relation we see that such motions are *isentropic*, i.e. they preserve constant entropy

$$\nabla s = 0 \ . \tag{3.39}$$

We must also specify boundary conditions to be satisfied by the velocity potential. Typically these will involve imposing the normal velocity along some surface. Along solid walls we would have

$$\vec{u} \cdot \vec{n} = \frac{d\phi}{dn} = 0 \tag{3.40}$$

which is known as the no through-flow condition. At computational interfaces typically the velocity is assumed to be given

$$\vec{u} \cdot \vec{n} = \frac{d\phi}{dn} = U_n \ . \tag{3.41}$$

Singularity methods in two dimensions.
Integral formulas for solutions to the Poisson equation. A typical problem one encounters in potential flow is determining the velocity field around some body $\Omega_b$ moving through a fluid - this is known as an *exterior flow problem*. Let us consider the simplest case first, that when the body is moving with a constant velocity $\vec{U}_\infty$. We can orient our $x$-axis parallel to this velocity direction and also choose to work in a reference frame attached to the body so that the body appears motionless and the fluid far away from the body's influence has the constant velocity $-U_\infty \vec{e}_x$ (note that this implies that the presence of the body induces perturbations which decay sufficiently fast away from the body, a point we shall return to later). We typically choose a computational domain of fluid around the body $\Omega = \Omega_\infty - \Omega_b$ such that on the far-field boundary of this computational domain $\Sigma_\infty = \Omega_\infty$ we have $\vec{U} \cong -U_\infty \vec{e}_x$ to whatever degree of precision required. The problem we wish to solve is therefore

$$\begin{aligned} \nabla^2 \phi &= 0 \text{ in } \Omega \\ \left.\frac{d\phi}{dn}\right|_{\Sigma_\infty} &= -\vec{U}_\infty \cdot \vec{n} \\ \left.\frac{d\phi}{dn}\right|_{\Sigma_b} &= 0 \end{aligned} \tag{3.42}$$

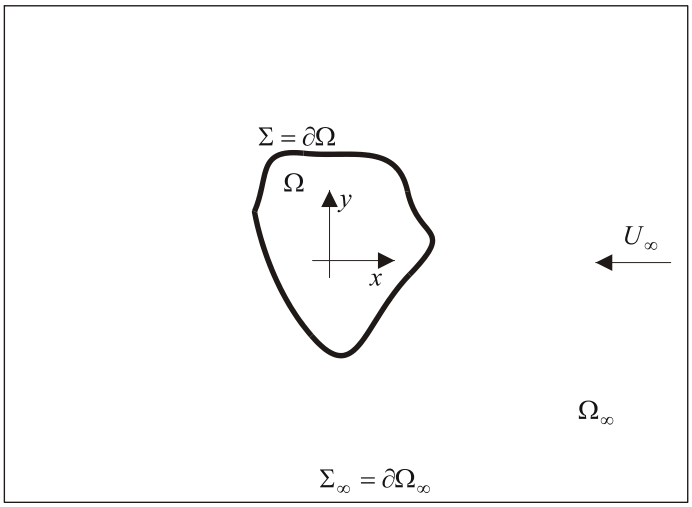
where the first boundary condition is given by the flow far away from the body and the second expresses no through-flow at the body surface.

Potential theory gives us analytical expressions for the solution that are useful computationally. For example, a function which is harmonic in $\Omega$ may always be expressed as an integral over the surfaces bounding $\Sigma = \partial\Omega$ as

$$\phi(P) = \frac{1}{4\pi} \int_\Sigma \left[ \frac{d\phi(M)}{dn} \left(\frac{1}{r}\right) - \phi(M) \frac{d}{dn} \left(\frac{1}{r}\right) \right] d\Sigma \ , \tag{3.43}$$

in three-dimensional space. For two dimensional flows this becomes

$$\phi(P) = \frac{1}{2\pi} \int_\Sigma \left[ \frac{d\phi(M)}{dn} \ln r - \phi(M) \frac{d}{dn} (\ln r) \right] d\Sigma \ . \tag{3.44}$$




Here $P$ denotes some point within $\Omega$ and $M$ a point on the surface $\Sigma$. The vector $\vec{r}$ is the relative position of $P$ from the viewpoint of $P$

$$\vec{r} = \vec{r}_P - \vec{r}_M \ , \tag{3.45}$$

and $r = |\vec{r}|$ so the two above formulas can be rewritten as

$$\phi(P) = \frac{1}{4\pi} \int_\Sigma \left[ \frac{d\phi(M)}{dn} \left( \frac{1}{|\vec{r}_P - \vec{r}_M|} \right) - \phi(M) \frac{d}{dn} \left( \frac{1}{|\vec{r}_P - \vec{r}_M|} \right) \right] d\Sigma_M \ \text{(3D)}, \tag{3.46}$$

$$\phi(P) = \frac{1}{2\pi} \int_\Sigma \left[ \frac{d\phi(M)}{dn} \ln |\vec{r}_P - \vec{r}_M| - \phi(M) \frac{d}{dn} \left( \ln |\vec{r}_P - \vec{r}_M| \right) \right] d\Sigma_M \ \text{(2D)}. \tag{3.47}$$

Relation to general operator theory methods. The above forms are examples of the general method of solving a linear differential equation through the fundamental solution technique, a topic we shall return to in some detail. In brief, if we wish to solve the general linear PDE

$$L\psi = \sigma \tag{3.48}$$

one approach is to first determine the *generalized fundamental solution* or *Green's function* $G$, i.e. the solution to the problem

$$LG = \delta \tag{3.49}$$

where $\delta$ is a Dirac delta distribution placed at the origin. Since the rhs involves distribution we shall say that $G$ is a generalized solution if it satisfies

$$(G, L^*\varphi) = (\delta, \varphi) \tag{3.50}$$

for some suitable space of trial functions $\varphi$. Here $L^*$ is the dual operator of $L$ and $(\cdot, \cdot)$ is some scalar product. The solution to the original problem can then be expressed as a convolution product

$$\psi = G * \sigma \ . \tag{3.51}$$

Written out explicitly the value of $\psi$ at some $x$ is

$$\psi(x) = \int G(x-y)\sigma(y)dy \ . \tag{3.52}$$

That the above formula is indeed a solution is easy to verify formally

$$L\psi = L\int G(x-y)\sigma(y)dy = \int LG(x-y)\sigma(y)dy = \int \delta(x-y)\sigma(y)dy = \sigma(x) \ . \tag{3.53}$$

Generally, solving the equation $L\psi = \sigma$ involves applying certain boundary conditions which lead to specific forms for the general integration operation shown above - these difficulties have been glossed over for now.

This general analytical procedure has many computational implementations. All of these depend on the ease with which the fundamental solution $G$ can be found. In general the Green's function also depends on the particular shape of the domain over which the operator $L$ is defined.

**Example 1.:** Consider as a first, simple example the initial value problem given by: $L = \frac{d}{dt} + 3$, $\sigma = e^{-2t}$

$$L\psi = \sigma \Leftrightarrow \frac{d\psi}{dt} + 3\psi = e^{-2t} \ , \tag{3.54}$$

with the initial condition $\psi(0) = 0$. We can easily find the solution by classical techniques to be

$$\psi(t) = e^{-2t} - e^{-3t} \ . \tag{3.55}$$

Let us rediscover this solution through the fundamental solution approach by first solving

$$LG = \delta. \tag{3.56}$$

We'll use the property

$$\frac{d\theta}{dt} = \delta \tag{3.57}$$

where $\theta$ is the Heaviside function, to quickly verify that the fundamental solution is

$$G(t) = \theta(t)e^{-3t} \tag{3.58}$$

since

$$G'(t) = \theta'(t)e^{-3t} - 3\ \theta(t)e^{-3t} \tag{3.59}$$

so that

$$LG = G'(t) + 3G(t) = \theta'(t)e^{-3t} - 3\ \theta(t)e^{-3t} + 3\ \theta(t)e^{-3t} = \delta(t)e^{-3t} = \delta(t) \ . \tag{3.60}$$

The last equality might seem a bit puzzling but recall that solutions are considered in a generalized sense so that $\delta(t)e^{-3t} = \delta(t)$ is understood as stating that

$$\left(\delta(t)e^{-3t}, \varphi\right) = (\delta(t), \varphi) \tag{3.61}$$

for some space of test functions $\varphi$ and this is true since

$$\int \delta(t)e^{-3t}\varphi(t)\ dt = \varphi(0) = \int \delta(t)\varphi(t)\ dt \ . \tag{3.62}$$

Having determined the fundamental solution we can immediately write down the solution to the initial value problem as

$$\psi = G * \sigma \Rightarrow \psi(t) = \int_0^\infty \theta(t-\tau) e^{-3(t-\tau)} e^{-2\tau}\ d\tau \Rightarrow \tag{3.63}$$

$$\psi(t) = \int_0^\infty \theta(t-\tau) e^{-3t} e^{\tau}\ d\tau = e^{-3t} \int_0^\infty \theta(t-\tau) e^{\tau}\ d\tau \tag{3.64}$$

$$= e^{-3t} \left[ \int_0^t \theta(t-\tau) e^{\tau}\ d\tau + \int_t^\infty \theta(t-\tau) e^{\tau}\ d\tau \right] \tag{3.65}$$

$$= e^{-3t} \left[ e^{\tau}|_{\tau=0}^{\tau=t} + 0 \right] = e^{-3t} \left( e^t - 1 \right) = e^{-2t} - e^{-3t}\ . \tag{3.66}$$

**Example 2.:** For $L = \frac{d^2}{dx^2} + a^2$ the fundamental solution is $G(x) = \theta(x) \frac{\sin ax}{a}$ so the solution to $\psi''(x) + a^2 \psi(x) = f(x)$ is

$$\psi(x) = \frac{1}{a} \int \theta(x-y) \sin a(x-y)\ f(y)\ dy\ . \tag{3.67}$$

Green's functions for the Laplace operator. For the Laplace equation the Green's function for an arbitrary domain $\Omega$ is typically hard to find, involving a complexity comparable to solving the full problem (3.42). Transforming the problem to an equivalent problem in free space is useful because we know the Green functions for $\mathbb{R}^2$ or $\mathbb{R}^3$ analytically for problems in which the solution decays to zero at infinity. In two dimensions, the solution to

$$\triangle G_2 = \delta \tag{3.68}$$

$$\lim_{r \to \infty} G_2 = 0 \tag{3.69}$$

is

$$G_2^{\triangle} = \frac{1}{2\pi} \log r = \frac{1}{4\pi} \log(x^2 + y^2)\ . \tag{3.70}$$

This corresponds to what is known as a *unit source singularity* placed at the origin, since the integral of the fluid flux over any curve that encloses the origin is a constant equal to 1 (for a fluid of unit density) and represents the amount of fluid injected into the domain by the point singularity. The elementary fluid flux is given by

$$dQ = \vec{u}\ d\vec{\Sigma}\ . \tag{3.71}$$

For the unit source singularity we have

$$\vec{u} = \nabla G_2^{\triangle} = \frac{\vec{r}}{2\pi r^2}\ . \tag{3.72}$$

Integrate this over a unit circle centered on the origin to obtain the total flux

$$Q = \int_0^{2\pi} \frac{\vec{r}}{2\pi r^2} r\ \frac{\vec{r}}{r} d\theta = 1 \tag{3.73}$$

where we used $d\vec{\Sigma} = r(\vec{r}/r)\ d\theta$, with $\vec{r}/r$ representing the outward pointing unit normal. It is easy to see that the integral over any other curve enclosing the origin would be the same (use integration in the complex plane and notice that the origin is a pole).

In three dimensions the solution to the analogous problem is

$$G_3^{\triangle} = -\frac{1}{4\pi} \frac{1}{r} = -\frac{1}{4\pi} \frac{1}{(x^2 + y^2 + z^2)^{1/2}}\ . \tag{3.74}$$

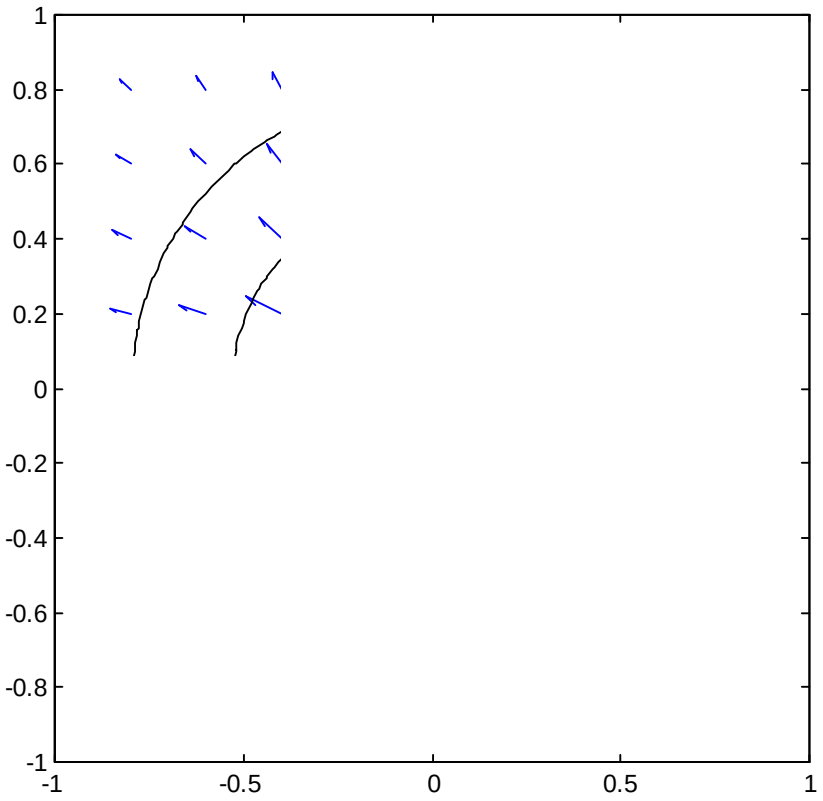


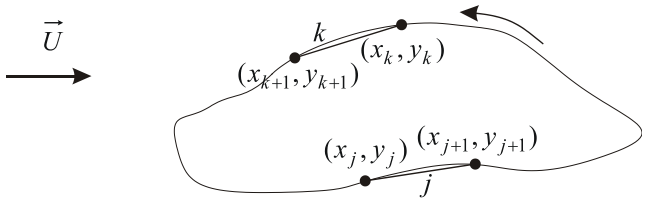

FIGURE 1. Flow field induced by a unit source singularity placed at the origin. also drawn are lines of constant potential.

Panel methods in 2D. The integral form of the solution can be used directly to generate useful numerical algorithms, a point that shall be investigated in conjunction with various fast summation algorithms we'll consider later on. To get some initial experience with integral equation methods for potential flow we'll consider one of the simplest, but still widely used, algorithms - the *panel method*.

Consider the 2D representation formula

$$\phi(P) = \frac{1}{2\pi} \int_\Sigma \left[ q(M) \ln r - \gamma(M) \frac{d}{dn} (\ln r) \right] d\Sigma \ . \tag{3.75}$$

One can read this as stating that the potential at some point is given by summing the contribution of the infinitesimal sources $q(M)d\Sigma$ and the infinitesimal dipoles $\gamma(M)d\Sigma$. A straightforward discretization suggested by this observation is to replace the curve $\Sigma$ by a piecewise linear approximation and to specify some simple function $q(M), \gamma(M)$ on each segment. The simplest case is when we take $q(M)$,$\gamma(M)$




constants over each segment thus leading to what is known as a constant intensity *source panel* or *dipole panel*, respectively. Global boundary conditions dictate whether both kinds of panels are needed, a detail we won't go into for now. Let us present a typical method based upon source panels only.

A 2D body is placed in a uniform current of velocity $\vec{U} = U\ \vec{e}_x$. The body's shape is approximated by a sequence of $N$ linear segments with a specified ordering of the panels. Each segment has a uniform source distribution of intensity $q_j$. These intensities are the unkowns of the problem and have to be determined from boundary conditions imposed on the body. Since boundary conditions on the body are given in terms of velocity (the no-through flow condition) we compute the velocity induced by all panels at a point $\vec{r}$ on the body's surface

$$\vec{V}(\vec{r}) = \vec{U} + \sum_{j=1}^{N} \vec{U}_j(\vec{r}) \tag{3.76}$$

where $\vec{U}_j(\vec{r})$ is the velocity induced by panel $j$ at position $\vec{r}$. To compute the induced velocity it is convenient to use a local coordinate system for each panel that has axis $\xi$ aligned with the panel. In this coordinate system the elementary velocity induced by the source $dQ = q_j d\xi$ at position $(\xi, 0)$ along the panel at a point $\vec{r}_k = \xi_k \vec{e}_\xi + \eta_k \vec{e}_\eta$ is given by

$$d\vec{U}_{jk} = \frac{q_j d\xi}{2\pi} \frac{\vec{r}_k - \xi \vec{e}_\xi}{|\vec{r} - \xi \vec{e}_\xi|^2} \tag{3.77}$$

Integration along the panel length from $\xi = -l_j/2$ to $\xi = l_j/2$ leads to

(3.78)

$$q_j\ \mu_{jk} = \vec{U}_{jk} \cdot \vec{e}_\xi = \frac{q_j}{2\pi} \int_{-l_j/2}^{l_j/2} \frac{(\xi_k - \xi)}{(\xi_k - \xi)^2 + \eta_k^2}\ d\xi = -\frac{q_j}{4\pi} \ln\left[(\xi - \xi_k)^2 + \eta_k^2\right]\Big|_{\xi=-l_j/2}^{\xi=l_j/2}$$

(3.79)

$$= -\frac{q_j}{4\pi} \ln \frac{(l_j/2 - \xi_k)^2 + \eta_k^2}{(l_j/2 + \xi_k)^2 + \eta_k^2}$$

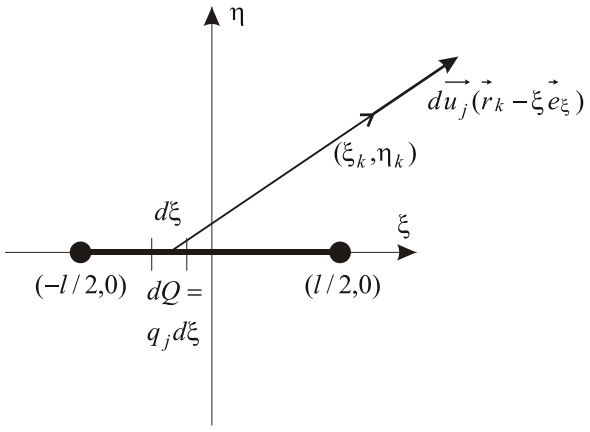
$$q_j\ \nu_{jk} = \vec{U}_{jk} \cdot \vec{e}_\eta = \frac{q_j}{2\pi} \int_{-l_j/2}^{l_j/2} \frac{\eta_k}{(\xi_k - \xi)^2 + \eta_k^2}\ d\xi \tag{3.80}$$

$$= -\frac{q_j}{2\pi} \left[\arctan \frac{\xi_k - l_j/2}{\eta_k} - \arctan \frac{\xi_k + l_j/2}{\eta_k}\right] \tag{3.81}$$

These velocities are easily transformed back into the global coordinate system

$$q_j\ u_{jk} = \vec{U}_{jk} \cdot \vec{e}_x = q_j\ \left[\mu_{jk}(\vec{e}_\xi \cdot \vec{e}_x) + \nu_{jk}(\vec{e}_\eta \cdot \vec{e}_x)\right] \tag{3.82}$$

$$q_j\ v_{jk} = \vec{U}_{jk} \cdot \vec{e}_y = q_j\ \left[\mu_{jk}(\vec{e}_\xi \cdot \vec{e}_y) + \nu_{jk}(\vec{e}_\eta \cdot \vec{e}_y)\right] \tag{3.83}$$




The no-through flow boundary condition is now imposed leading to a linear system for the unknown panel source intensities $q_j$

$$\vec{V}(\vec{r}_k) \cdot \vec{n}_k = 0 = \vec{U} \cdot \vec{n}_k + \sum_{j=1}^{N} \vec{U}_j(\vec{r}_k) \cdot \vec{n}_k \tag{3.84}$$

where $\vec{n}_k$ is the unit vector normal to panel $k$. The explicit form of the system is

$$\sum_{j=1}^{N} (u_{jk} n_{k,x} + v_{jk} n_{k,y}) \;\; q_j = -\vec{U} \cdot \vec{n}_k \tag{3.85}$$

Complex variable techniques. Complex function methods are extremely useful in studying two-dimensional potential (inviscid, incompressible) flows with no initial vorticity. Recall that for these types of flows the velocity field can be expressed as a scalar potential

$$\vec{V} = \nabla \phi \ , \tag{3.86}$$

and that the potential function is harmonic because of the incompressibility constraint $\nabla \cdot \vec{V} = 0$ arising from the continuity equation

$$\triangle \phi = 0 \ . \tag{3.87}$$

Componentwise we have

$$\vec{V} = u\vec{e}_x + v\vec{e}_y \tag{3.88}$$

$$u = \phi_x, \; v = \phi_y \tag{3.89}$$

and since $\phi$ is harmonic

$$u_x = -v_y \ . \tag{3.90}$$

A curve to which the velocity vector is always tangent is called a streamline and determined by

$$\frac{dx}{u} = \frac{dy}{v} \ . \tag{3.91}$$

Note that the above relation can be expressed as

$$-v \; dx \; + u \; dy = 0 \ . \tag{3.92}$$

This a differential expression

$$A(x,y)dx + B(x,y)dy \tag{3.93}$$

which may be expressed as the total differential of some function $\psi$ if $A_y = B_x$ or in this particular case

$$-v_y = u_x \ . \tag{3.94}$$

This is exactly the incompressibility constraint $u_x + v_y = 0$ so there does exist a total (exact) differential $\psi$ with the properties

$$d\psi = -v\ dx\ + u\ dy, \tag{3.95}$$

$$\psi_x = -v,\ \psi_y = u \ . \tag{3.96}$$

Furthermore, along streamlines we see that

$$d\psi = 0 \ . \tag{3.97}$$

Since the fluid is assumed to be irrotational, the component of the vorticity $\nabla \times \vec{V}$ perpendicular to the $xy$-plane must be zero

$$v_x - u_y = 0 \ , \tag{3.98}$$

which implies

$$\psi_{xx} + \psi_{yy} = 0 \ , \tag{3.99}$$

so the function $\psi$ is also harmonic.

We know that holomorphic functions in the complex $z = x + iy$ plane have harmonic real and imaginary parts so we can define a complex potential

$$f(z) = \phi(x,y) + i\psi(x,y) \ . \tag{3.100}$$

Since $f$ is holomorphic we have

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \tag{3.101}$$

$$\frac{\partial f}{\partial x} = \phi_x + i\psi_x = u - iv \tag{3.102}$$

$$-i\frac{\partial f}{\partial y} = -i\left(\phi_y + i\psi_y\right) = u - iv \tag{3.103}$$

so we see that the derivative of the complex potential leads to quantity similar to the velocity field vector, albeit with a change of sign in the $y$-component. We shall call this quantity the complex velocity

$$w = \frac{df}{dz}. \tag{3.104}$$

The conjugate complex velocity $\bar{w}$ is equivalent to the velocity vector $\vec{V}$.

3.1.4. *Inviscid, compressible flow - Euler equations of gas dynamics.*

One-dimensional flow.

Eigenstructure of the 1D Euler equations. In the absence of viscosity the conservation form of the fluid dynamic equations in 1D is

$$q_t + f(q)_x = 0 \tag{3.105}$$

$$q = \begin{bmatrix} \rho & \rho u & \rho E \end{bmatrix}^T \tag{3.106}$$

$$f = \begin{bmatrix} \rho u & \rho u^2 + p & \rho u H \end{bmatrix}^T \tag{3.107}$$

Faced with the task of devising a numerical method to solve the above system of equations, our first goal is to determine the type of PDE system we have. We do this by computing the eigenvalues of the local linearization

$$q_t + f_q q_x = 0 \tag{3.108}$$

with the coefficients within the Jacobian matrix $f_q$ assumed to be frozen (this is the linearization). Though one could directly compute the eigenstructure of $f_q$ it is typically more convenient to work with the primitive variables

$$w = \begin{bmatrix} \rho & u & p \end{bmatrix}^T \ . \tag{3.109}$$

Through the general equations for an gas we know that a relationship $q(w)$ exists and physical considerations imply the existence of an inverse also $w(q)$. The conservation equations can be rewritten as

$$q_w w_t + f_q q_w w_x = 0 \tag{3.110}$$

or

$$w_t + A w_x = 0 \tag{3.111}$$

with

$$A = (q_w)^{-1} f_q q_w \ . \tag{3.112}$$

Note that the above relation is a similarity transform from the conservative variable Jacobian $f_q$ to a new matrix $A$. The form of $A$ is most easily determined by replacing $q$ with $w$ and applying thermodynamic relations in the original conservative system. The conservation form of the continuity equation

$$\rho_t + (\rho u)_x = 0 \tag{3.113}$$

immediately leads to the primitive variable form

$$\rho_t + u\rho_x + \rho u_x = 0 \ . \tag{3.114}$$

The conservative momentum equation

$$(\rho u)_t + (\rho u^2 + p)_x = 0 \tag{3.115}$$

likewise gives

$$u([\rho_t + (\rho u)_x] + \rho u_t + \rho u u_x + p_x = 0 \tag{3.116}$$

with the quantity in the brackets being null by the conservative continuity equation. The energy equation is the only equation that requires a bit more work. The

equations

$$(\rho E)_t + (\rho u H)_x = 0 \tag{3.117}$$

$$E = \frac{1}{\gamma - 1}\frac{p}{\rho} + \frac{u^2}{2} \tag{3.118}$$

$$H = E + \frac{p}{\rho},\ a^2 = \gamma\frac{p}{\rho} \tag{3.119}$$

lead to

$$p_t + u p_x + \gamma p u_x = 0. \tag{3.120}$$

The final form we obtain for $A$ is therefore

$$A = \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho a^2 & u \end{bmatrix}. \tag{3.121}$$

The eigenvalues are given by

$$\det(A - \lambda I) = (u - \lambda)\left[(u - \lambda)^2 + a^2\right] = 0 \tag{3.122}$$

leading to

$$\lambda_1 = u + a,\ \lambda_2 = u,\ \lambda_3 = u - a\ . \tag{3.123}$$

The associated eigenmode matrix is

$$R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} = \begin{bmatrix} \rho & 1 & \rho \\ a & 0 & -a \\ \rho a^2 & 0 & \rho a^2 \end{bmatrix}\ . \tag{3.124}$$

The determinant of $R$ is

$$\det R = -2a^3\rho \tag{3.125}$$

which is nonzero since $a$ and $\rho$ are physical, positive quantities so the eigenvectors form a basis for 3-space and the primitive variable system $w_t + A w_x = 0$ is hyperbolic. Because of the existence of the similarity transform, the original conservative variable system is also hyperbolic.

Since the system is hyperbolic, we know that one way of seeking a solution is by marching along characteristic curves. In preparation for this let us determine the characteristic form of the equation. We write

$$A = R\Lambda R^{-1} \tag{3.126}$$

with

$$\Lambda = \begin{bmatrix} u + a & & \\ & u & \\ & & u - a \end{bmatrix} \tag{3.127}$$

$$R^{-1} = \frac{1}{2a^2}\begin{bmatrix} 0 & a & 1/\rho \\ 2a^2 & 0 & -2 \\ 0 & -a & 1/\rho \end{bmatrix} \tag{3.128}$$

The primitive variable system is therefore

$$w_t + R\Lambda R^{-1} w_x = 0 \tag{3.129}$$

leading to the characteristic system

$$z_t + \Lambda z_x = 0 \tag{3.130}$$

with the characteristic variables

$$z = R^{-1}w = \begin{bmatrix} \frac{1}{2a}u + \frac{1}{2a^2}\frac{p}{\rho} \\ \rho - \frac{1}{a^2}p \\ -\frac{1}{2a}u + \frac{1}{2a^2}\frac{p}{\rho} \end{bmatrix} \tag{3.131}$$

Note that up to now we've assumed that the system was linearized by freezing the local values of coeeficients of the derivatives. This can be motivated as a pertubation expansion of the flow variables

$$w = \bar{w} + w' \tag{3.132}$$

in which $\bar{w}$ is some average state that corresponds to the frozen coefficients and $w^{'}$ are perturbations around this state. If the perturbations are small we have

$$(\bar{w} + w')_t + (\bar{A} + A')(\bar{w} + w')_x = 0 \tag{3.133}$$

which gives

$$w'_t + \bar{A}w'_x = 0 \tag{3.134}$$

since derivatives of $\bar{w}$ are zero, and neglecting second order quantitites (i.e. $A'w'_x$). The small-perturbation characteristic variables are now

$$z' = \bar{R}^{-1}w' = \frac{1}{2\bar{a}^2}\begin{bmatrix} 0 & \bar{a} & 1/\bar{\rho} \\ 2\bar{a}^2 & 0 & -2 \\ 0 & -\bar{a} & 1/\bar{\rho} \end{bmatrix}\begin{bmatrix} \rho' \\ u' \\ p' \end{bmatrix} = \begin{bmatrix} \frac{1}{2\bar{a}}u' + \dfrac{1}{2\bar{a}^2\bar{\rho}}p' \\ \rho' - \frac{1}{\bar{a}^2}p' \\ -\frac{1}{2\bar{a}}u' + \dfrac{1}{2\bar{a}^2\bar{\rho}}p' \end{bmatrix} \tag{3.135}$$

an entity of considerable physical importance. Recall that for a linear system of hyperbolic variables, the characteristic variables remain constant along their respective characteristic curves. For the non-linear system of 1D Euler equations this is no longer strictly the case, but for small perturbations components of $z'$ remain constant along the respective characteristic curves. Hence, small perturbations propagate in accordance with one of the three modes above. The physical nature of the above waves is determined by applying thermodynamic relations; we will skip the details and just list the physical nature of the eigenmodes:

(1) $\pm u'/(2\bar{a}) + p'/(2\gamma\bar{p})$ corresponds to the propagation of sound waves with velocity $\bar{u} \pm \bar{a}$
(2) $\rho' - p'/\bar{a}^2$ corresponds to the propagation of entropy waves with the flow velocity $u$

The Riemann problem for 1D Euler equations. A number of numerical methods for hyperbolic PDE's rely on the solution to a specific initial-value problem

$$q(x, t = 0) = \begin{cases} q_l & x < 0 \\ q_r & x > 0 \end{cases} \tag{3.136}$$

known as the Riemann problem. To solve a system of linear hyperbolic PDE's with the above initial condition we would expand the jump in $q$ at the origin on the basis formed by the eigenvectors of the system matrix

$$q_r - q_l = \sum_{k=1}^{m} \alpha_k r_k \tag{3.137}$$

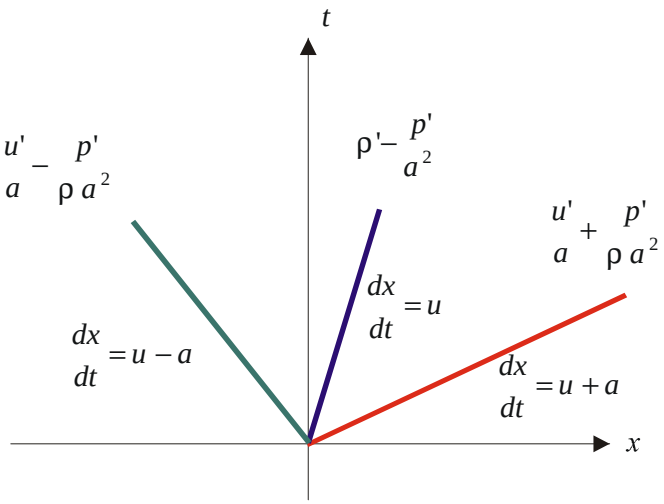



FIGURE 2. Typical geometry of characteristic curves for the 1D Euler equations showing the slopes and Riemann invariants.

and then follow the propagation of each eigenmode independently of the others. This is no longer possible for non-linear systems since pasage of the perturbations associated with one eigenmode affects all the other modes.

Nonetheless, there exist some systems, the representative example of which is the Euler equations, for which a complete solution for the Riemann problem is available. The crucial aspect is that there exist quantities that remain constant along characteristic curves, somewhat similar to how characteristic variables remain constant along characteristic lines for linear problems. These are called Riemann invariants and the Riemann invariants for the 1D Euler equations are

(1) along sound propagation characteristics

$$s,\ u \pm \frac{2a}{\gamma - 1} \tag{3.138}$$

(2) along the entropy propagation characteristic

$$u,\ p \tag{3.139}$$

Note that there are two quantities that remain constant along each characteristic curve. One can plot the surfaces defined by constant values of the Riemann invariants in $(\rho, u, p)$ space and seek intersections between these families that correspond to finding a physical path between $(\rho_l, u_l, p_l)$ and $(\rho_r, u_r, p_r)$.

Godunov type methods for the Euler equations. The knowledge of the characteristic structure of the equations and of the solution to the Riemann problem makes it straightforward to construct a Godunov type method for the Euler equations. Let us consider a finite volume method. The same ideas can be implemented also in a finite difference or a finite element context. We introduce a grid $\{a = x_0, x_1 ..., x_M = b\}$ partitioning the domain $[a, b]$ into finite volume cells $C_j = [x_{j-1}, x_j]$. Integrating the conservative form of the Euler equations over a time step $[t^n, t^{n+1}]$ and over a

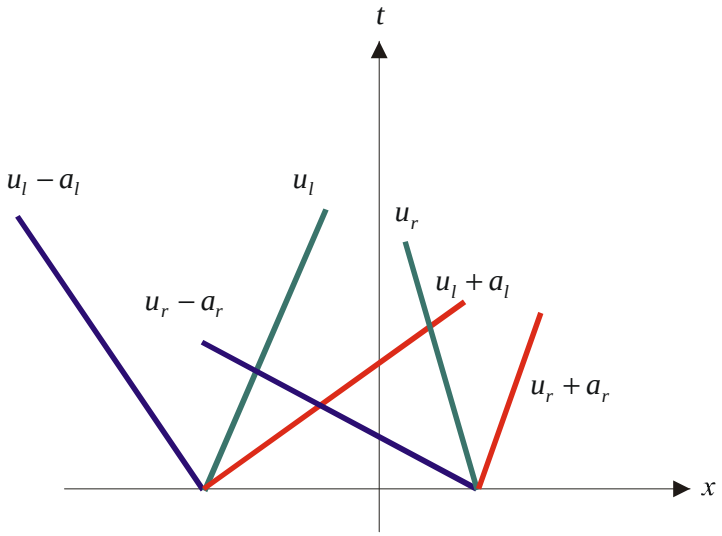



FIGURE 3. Typical aspect of characterisitic lines near the discontinuity at the origin in a Riemann problem.

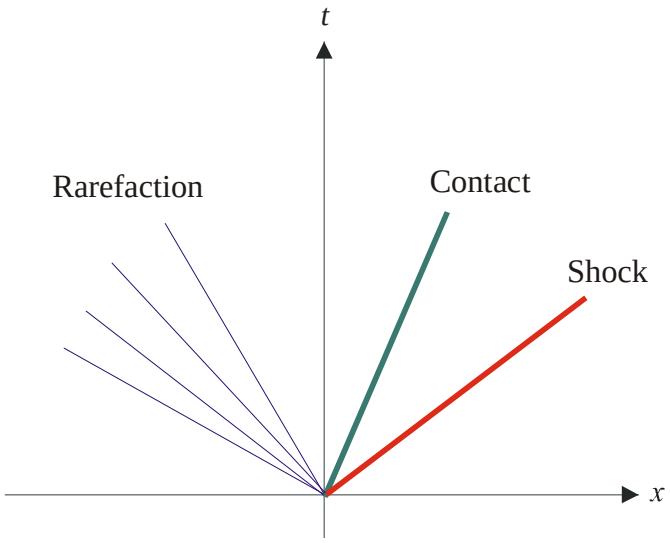



FIGURE 4. Typical pattern of discontinuities formed by interaction of the characteristics from the left and right states in the Riemann problem for the 1D Euler equations.

cell $C_j$ leads to

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\Delta x}\left(F_j - F_{j-1}\right) \tag{3.140}$$

with

$$Q_j^n = \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} q(x,t^n)dx,\ F_j = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(q(x_j,t))\ dt. \tag{3.141}$$

The main problem faced in constructing a Godunov method is to establish a procedure to compute the numeric fluxes $F_j$.