Overview of finite difference methods and analysis

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$$\sum_{j=0}^{r} a_j Q^{(n+j)} = k \sum_{j=0}^{r} b_j f^{n+j}$$

k = time step size

$$\boxed{u' = f(t, u)}, \quad u(t)$$

$$Q^{(n+j)} \approx u(t^{(n+j)})$$

$$t^{n+j} = (n+j)k$$

f Euler

$$Q^{n+1} = Q^{n} + kF^{n}$$

$$a_0 = -1 \qquad a_1 = 1$$

$$\rho(\zeta) = \sum_{j}^{r} a_j \zeta^{j} \qquad \rho(1) = 0$$

$$\rho(\zeta) = \sum_{j=0} a_j \zeta^j \qquad \rho(1) = 0$$

$$\sigma(\zeta) = \sum_{j=0}^{r} b_j \zeta^j \qquad b_0 = 1$$

Trapezoid

$$\varrho^{n+1} = \varrho^{n} + \frac{k}{2}\left(F^{n} + F^{n+1}\right)$$

$$F^{n} \equiv f(t^{n}, \varrho^{n})$$

$$\rho(\zeta) = \zeta - 1 \; ; \; \rho(1) = 0$$

$$\sigma(\zeta) = \frac{1}{2}\zeta + \frac{1}{2} \; ; \; \sigma$$

$$\tilde{\Delta} = \frac{1}{k}\sum_{j=0} a_j E^j \xrightarrow[(k \to 0)]{} \frac{d}{dt}$$

$$D = \sum_{j=0}^{r} b_j f(E^j) \underset{r \to \infty}{\longrightarrow} f$$

$$E^j$$

$$\tau^n = (\tilde{D} - D)\, g(t^n)$$

Consistency

$$\boxed{\begin{gathered} \rho(1) = 1 \\ \rho'(1) = \sigma(1) \end{gathered}}$$

$$\pi(\zeta;z) = \rho(\zeta) - z\,\sigma(\zeta)$$

$$z = k\lambda$$

$y' = f(t, y)$ → Model problem

$$y' = \lambda y$$

$$\lambda \cong \frac{\partial f}{\partial y}$$

Euler

$$y^{n+1} - y^n = z\,y^n$$

$\rho(\zeta) = \zeta - 1 \quad \sigma(\zeta) = 1$

$$\pi(\zeta;z) = \rho(\zeta) - z\,\sigma(\zeta)$$

$= \zeta - 1 - z = 0$

$\Rightarrow \zeta = 1 + z$

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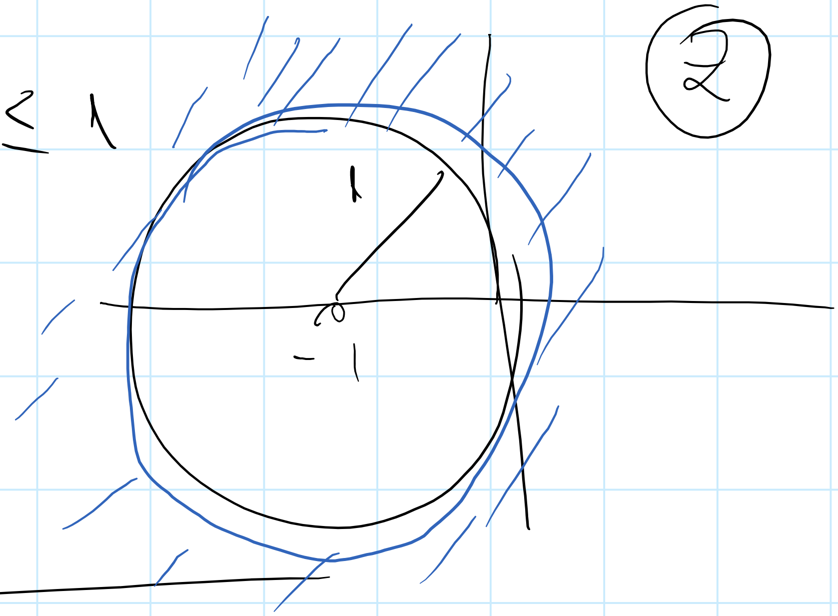
$|1+z| \le 1$



$$\rho^{n+1} = \rho^{n} + z\rho^{n+1}$$

$$\rho(\zeta) = \zeta - 1$$

$$\sigma(\zeta) = \zeta$$

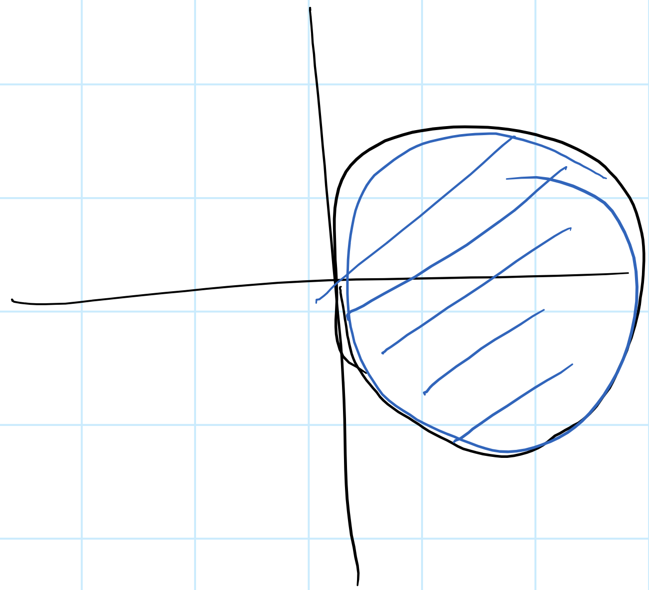
$$\pi(\zeta; z) = \zeta - 1 - z\zeta = 0$$

$$\zeta = \frac{1}{1-z}$$

$$|\zeta| = \frac{1}{|1-z|} \leq 1 \Rightarrow$$

$$|1 - z| \geq 1$$

z



Trapezoid

$$\rho(\zeta) = \zeta - 1$$

$$\sigma(\zeta) = \frac{1}{2}(\zeta + 1)$$

$$\pi(\zeta; z) = \zeta - 1 - \frac{z}{2}(\zeta + 1) = 0$$

$$\left(1-\frac{z}{2}\right)J = 1+\frac{z}{2} \Rightarrow$$

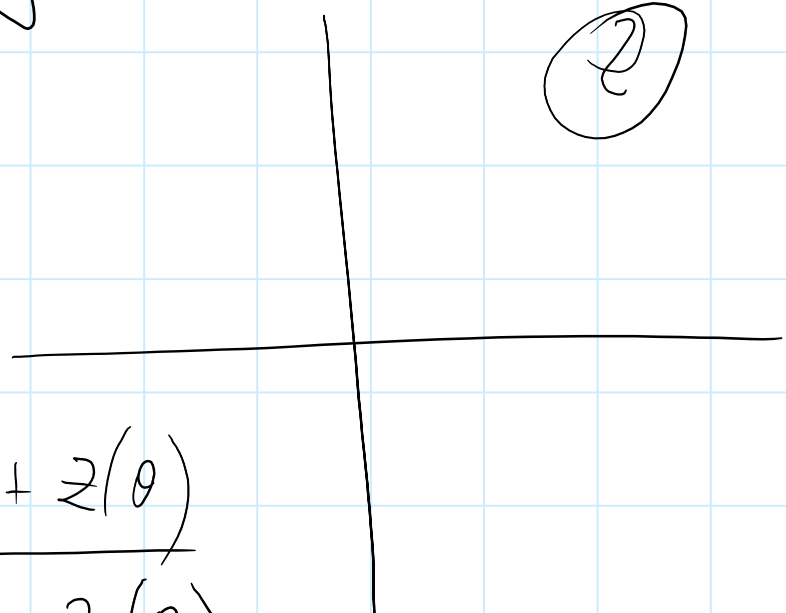
$$J = \frac{2+z}{2-z}$$

$z = \lambda k$

Boundary locus

z

$$J = e^{i\theta}$$



$$e^{i\theta} = \frac{2+z(\theta)}{2-z(\theta)}$$

$$2e^{i\theta} - e^{i\theta}z(\theta) = 2 + z(\theta)$$

$$z(\theta)\left[1+e^{i\theta}\right] = 2\left(e^{i\theta}-1\right)$$

$$z(\theta) = 2\,\frac{e^{i\theta}-1}{e^{i\theta}+1}$$

$$0 \le \theta < 2\pi$$

$$= \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{-(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2})}$$

$$z(\theta) = 2\,\frac{\cos\theta - 1 + i\sin\theta}{\cos\theta + 1 + i\sin\theta}$$

$$= 2\,\frac{-2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= 4\tan\frac{\theta}{2}\cdot\frac{-\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}}{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}$$

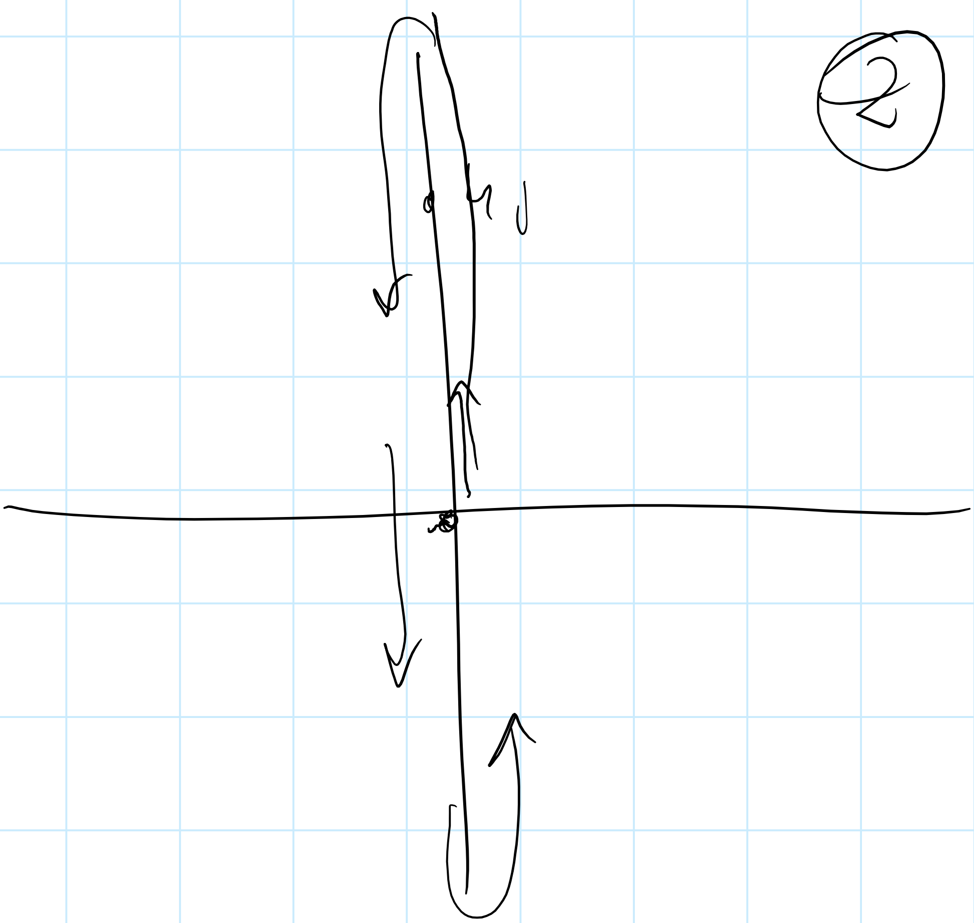
$$= 4\tan\frac{\theta}{2}\,\frac{\cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)}{e^{i\theta/2}}$$

$$e^{i\frac{\pi}{2}} \quad e^{i\frac{\theta}{2}}$$

$$= 4 \tan\frac{\theta}{2} \; \frac{e^{i\frac{\pi}{2}} \cancel{e^{i\frac{\theta}{2}}}}{\cancel{e^{i\theta/2}}} =$$

$$= 4i \tan\frac{\theta}{2}$$

2



$$\theta = 0 \Rightarrow z(0) = 0$$

$$\theta = \frac{\pi}{2} \Rightarrow z\left(\frac{\pi}{2}\right) = 4i$$

$\theta = \tau \Rightarrow Z(\pi_1)$