Overview of finite difference methods and analysis

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$$\sum_{j=0}^{r} a_j U^{(n+j)} = k \sum_{j=0}^{r} b_j f^{n+j}$$

k = time step size

$$\boxed{u' = f(t, u)}, \; u(t)$$

$$U^{(n+j)} \approx u(t^{(n+j)})$$

$$t^{n+j} = (n+j)k$$

f Euler

$$U^{n+1} = U^{n} + kF^{n}$$

$$a_0 = -1 \quad a_1 = 1$$

$$\rho(\zeta) = \sum_{j=0}^{r} a_j \zeta^j \qquad \rho(1) = 0$$

$$\sigma(\zeta) = \sum_{j=0}^{r} b_j \zeta^j \qquad b_0 = 1$$

Trapezoid

$$U^{n+1} = U^{n} + \frac{k}{2}\left(F^{n} + F^{n+1}\right)$$

$$F^{n} \equiv f(t^{n}, U^{n})$$

$$\rho(\zeta) = \zeta - 1 \;;\; \rho(1) = 0$$

$$\sigma(\zeta) = \frac{1}{2}\zeta + \frac{1}{2} \; ; \; \sigma$$

$$\tilde{D} = \frac{1}{k}\sum_{j=0}^{r} a_j E^j \xrightarrow[(k\to 0)]{} \frac{d}{dt}$$

$$D = \sum_{j=0}^{r} b_j f(E^j) \xrightarrow[k\to\infty]{} f$$

$$E^j$$

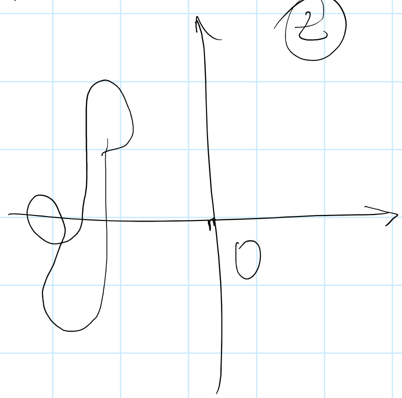
$$\tau^n = (\tilde{D} - D)\, v(t^n)$$

Consistency

$$z(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}$$

$$\boxed{\begin{aligned} \rho(1) &= 0 \\ \rho'(1) &= \sigma(1) \end{aligned}}$$

z



$$\pi(\zeta; z) = \rho(\zeta) - z\sigma(\zeta)$$

$$z = k\lambda$$

$v' = f(t, v)$ → Model problem

$$v' = \lambda v$$

$$\lambda \cong \frac{\partial f}{\partial v}$$

$$\lambda \cong \frac{\partial f}{\partial \zeta}$$

Euler

$$\phi^{n+1} - \phi^{n} = z\phi^{n}$$

$$\rho(\zeta) = \zeta - 1 \qquad \sigma(\zeta) = 1$$

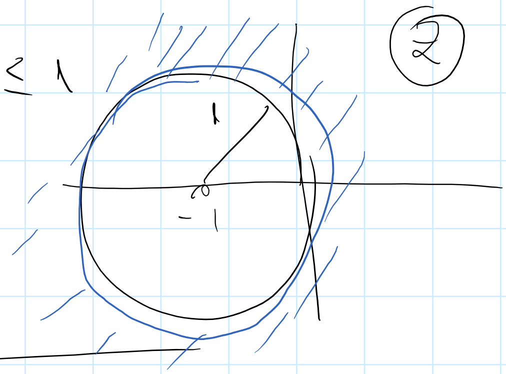
$$\pi(\zeta; z) = \rho(\zeta) - z\,\sigma(\zeta)$$

$$= \zeta - 1 - z = 0$$

$$\Rightarrow \zeta = 1 + z$$

$$|1+z| \le 1$$

2



$$\phi^{n+1} = \phi^{n} + z\phi^{n+1}$$

$$\rho(\zeta) = \zeta - 1$$

$$\sigma(\zeta) = \zeta$$

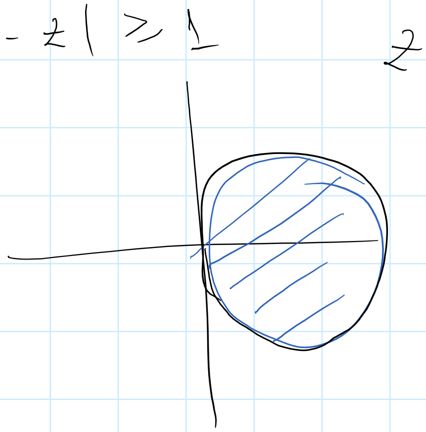
$$\pi(\zeta; z) = \zeta - 1 - z\zeta = 0$$

$$\zeta = \frac{1}{1-z}$$

$$|\zeta| = \frac{1}{\quad} < 1 \quad -1$$

$|1 - z| = 1 \ldots$

$|1 - z| \geq 1$



Trapezoid

$\rho(\zeta) = \zeta - 1$

$\sigma(\zeta) = \frac{1}{2}(\zeta + 1)$

$\tilde{\pi}(\zeta; z) = \zeta - 1 - \frac{z}{2}(\zeta + 1) = 0$

$\left(1 - \frac{z}{2}\right)\zeta = 1 + \frac{z}{2} \Rightarrow$

$\zeta = \frac{2 + z}{2 - z}$

$\tilde{\pi}(\zeta; 1) =$

$\zeta - 1 - \frac{\zeta}{2} - \frac{1}{2} = 0$

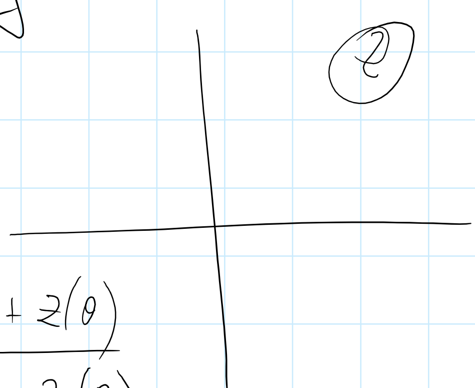
$\frac{\zeta}{2} = \frac{3}{2} \Rightarrow$

$\zeta = 3$

$z = \lambda k$

Boundary locus

$\zeta = e^{i\theta}$



$e^{i\theta} = \frac{2 + z(\theta)}{2 - z(\theta)}$

$y' = \lambda y$

$z = \lambda k$

$y' = \lambda y + s(t)$

$2 \cdot z(\theta)$ $\quad$ 1

$$2e^{i\theta} - e^{i\theta} z(\theta) = 2 + z(\theta)$$

$$z(\theta)\left[1 + e^{i\theta}\right] = 2\left(e^{i\theta} - 1\right)$$

$$z(\theta) = 2\,\frac{e^{i\theta} - 1}{e^{i\theta} + 1}$$

$0 \le \theta < 2\pi$

$$= \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{-\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)}$$

$$z(\theta) = 2\,\frac{\cos\theta - 1 + i\sin\theta}{\cos\theta + 1 + i\sin\theta}$$

$$= 2\,\frac{-2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

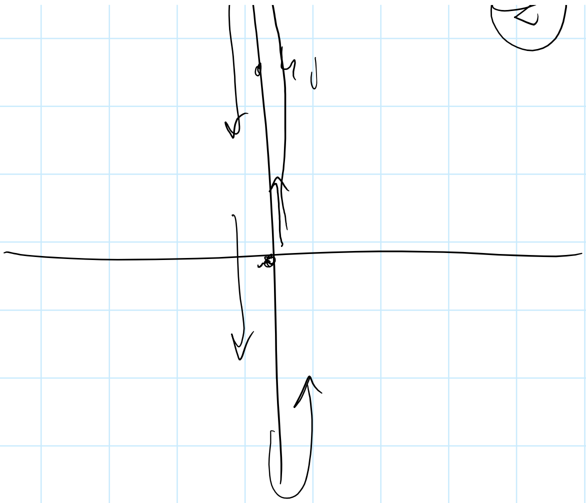
$$= 4\tan\frac{\theta}{2} \cdot \frac{-\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}}{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}$$

$$= 4\tan\frac{\theta}{2}\,\frac{\cos\left(\frac{\pi}{2} + \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right)}{e^{i\theta/2}}$$

$$= 4\tan\frac{\theta}{2}\,\frac{e^{i\frac{\pi}{2}}\cancel{e^{i\frac{\theta}{2}}}}{\cancel{e^{i\theta/2}}} =$$

$$= 4i\tan\frac{\theta}{2}$$

2

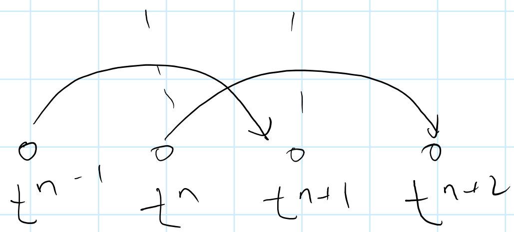


$$\theta = 0 \Rightarrow z(0) = 0$$
$$\theta = \frac{\pi}{2} \Rightarrow z\left(\frac{\pi}{2}\right) = 4i$$
$$\theta = \pi \Rightarrow z(\pi)$$

$$\pi(\zeta; z) = \underline{\rho(\zeta)} - z\,\underline{\sigma(\zeta)} \qquad z = \lambda k$$

Model problem $\begin{cases} u' = \lambda u \\ u(0) = u_0 \end{cases}$

Leapfrog

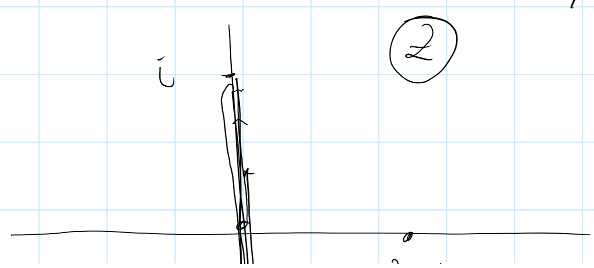


$$u^{n+1} = u^{n-1} + 2z\,u^{n}$$

$$\rho(\zeta) = \zeta^2 - 1 \qquad \sigma(\zeta) = \zeta$$

$$\zeta = e^{i\theta} \qquad \pi(e^{i\theta}; z) = 0 \qquad z(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}$$

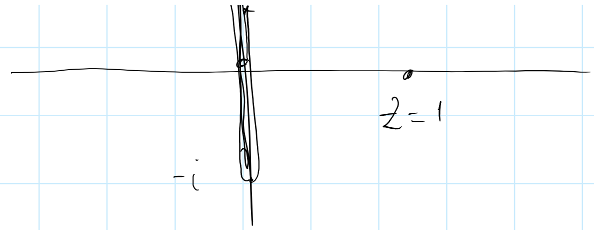
$$z(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2} = i \sin\theta$$



$$\pi(\zeta; 1) = \zeta^2 - 1 - \zeta = 0$$

$$\Pi(\zeta;1) = \zeta^2 - 1 - \zeta = 0$$

$$\Rightarrow \quad \zeta_{1,2} = \frac{1 \pm \sqrt{1+4}}{2}$$



$$\rho_t + u\rho_x = 0 \qquad \text{advection equation}$$

$$\frac{d\rho}{dt} = \frac{u}{2h} A \rho \qquad \rho(t) = \begin{pmatrix} \rho_1(t) \\ \rho_2(t) \\ \vdots \\ \rho_m(t) \end{pmatrix}$$

$$\rho_i(t) \cong \rho(x_i, t) \qquad x_i = ih$$

$$A = \begin{pmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & -1 & 0 & 1 \\ & & & 0 \end{pmatrix} = \mathrm{diag}([-1 \quad 0 \quad 1])$$

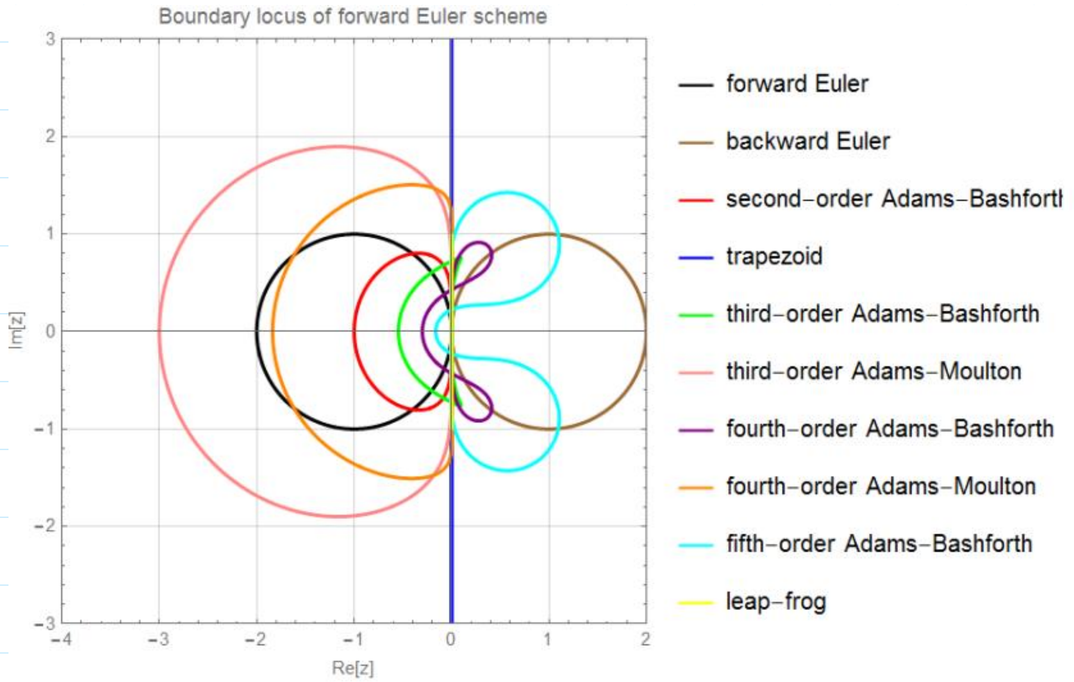
Adams-Bashforth of order 2

$$\rho^{n+2} = \rho^{n+1} + \frac{k}{2}\left(3\rho^{n+1} - \rho^{n}\right)$$

$$\rho^{n+2} = \rho^{n+1} + \frac{k}{2}\left(3 f(\rho^{n+1}) - f(\rho^{n})\right)$$

Adams-Moulton of order 3

See BoundaryLocus.nb on how to generate a famous plot in ODE numerical methods


Boundary locus of forward Euler scheme
Im[z]
Re[z]
3
2
1
0
−1
−2
−3
−4
−3
−2
−1
0
1
2
forward Euler
backward Euler
second−order Adams−Bashfortl
trapezoid
third−order Adams−Bashforth
third−order Adams−Moulton
fourth−order Adams−Bashforth
fourth−order Adams−Moulton
fifth−order Adams−Bashforth
leap−frog