Solution by characteristics of first-order PDEs

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First-order scalar PDE

(1) $F(\rho_t, \rho_x, \rho) = 0$

Implicit function theorem

$F(x,y) = 0 \Rightarrow y(x)$

if $F \in C^1$

$$\frac{dy}{dx} =$$

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

$$\rho_t + f(\rho_x) = 0$$

$$\rho : [0,1] \times [0,\infty) \to \mathbb{R}$$

$$\begin{cases} \rho_t + u\rho_x = 0 \\ \rho(x, t=0) = \alpha(x) \\ \rho(x=0, t) \end{cases}$$

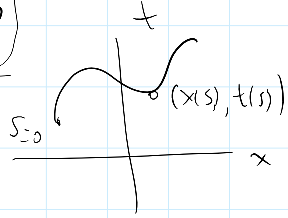
$$\rho(x,t) =$$

Solution by characteristics

$$\Big\{ \rho_t + u\rho_x = e^t$$

Solution by characteristics

$$\boxed{\rho_t + u\rho_x = 0}$$



$$\Gamma : \begin{cases} x(s) \\ t(s) \end{cases}$$

$$\rho(x,t)\Big|_\Gamma = \rho(x(s), t(s))$$

$$d\rho\Big|_\Gamma = \frac{\partial\rho}{\partial x}\frac{dx}{ds} + \frac{\partial\rho}{\partial t}\frac{dt}{ds} = 0$$

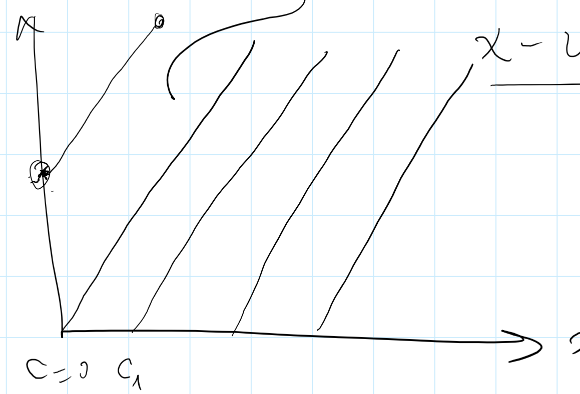
$$\frac{dt}{ds} = 1 \qquad \frac{dx}{ds} = u$$

$$\rho_t + u\rho_x = e^t$$

$$s = t$$

$$\boxed{\frac{d\rho}{dt}\Big|_\Gamma = e^t \Rightarrow}$$

$$\rho(\underbrace{ut+c}_{}, t) = A + e^t$$



$$x - ut = c$$

$$\rho(c, 0) = A + e^t =$$

$$= A + 1 = \sin c$$

$$\rho(x,t) = \sin(x - ut) + e^t$$

$$\rho_t = -u\cos(x - ut) + e^t$$

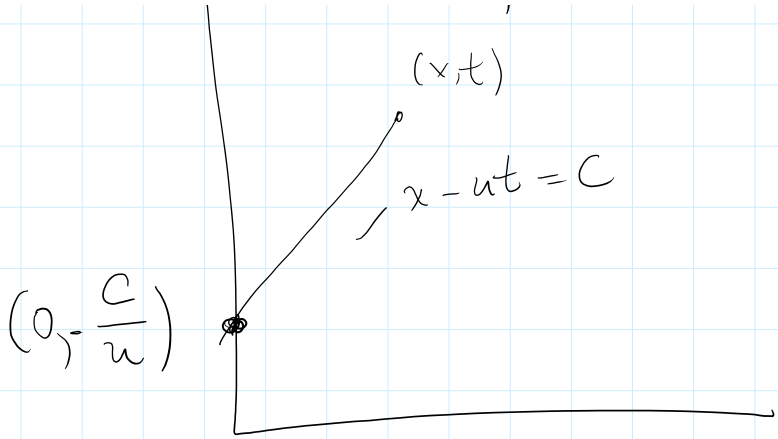
$$u\rho_x = u\cos(x - ut)$$

$$= e^t$$

$$\rho(x,t) = \begin{cases} \sin(x - ut) + e^t - 1 & x - ut \geq 0 \\ \cos\left(\frac{x - ut}{u}\right) + e^t & x - ut < 0 \\ & x - ut < 0 \end{cases}$$

(x,t)

$$\rho\left(0, -\frac{c}{u}\right) = A + e^t$$



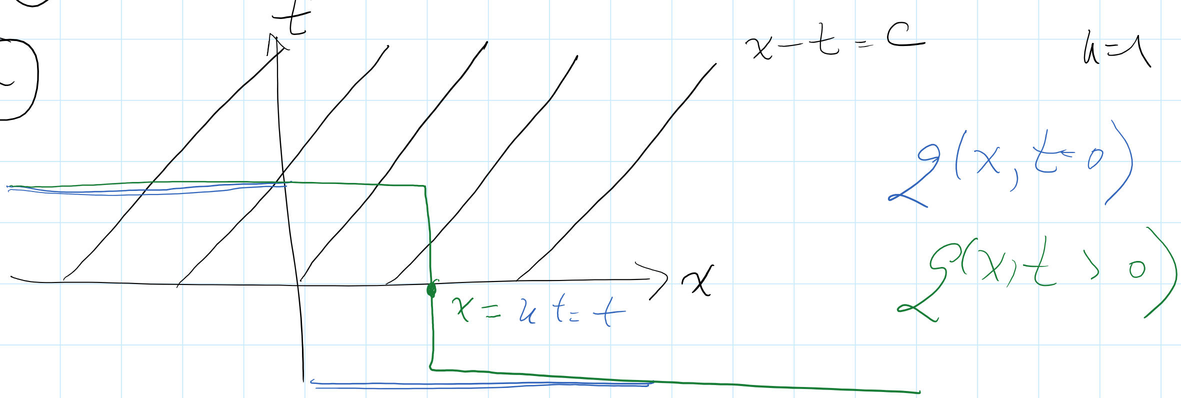
$$\rho\left(0, -\frac{C}{u}\right) = A + e^t$$
$$= \cos\frac{C}{u} + e^t$$
$$\rho(x,t) = \cos\frac{x-ut}{u} + e^t$$

$$\begin{cases} \rho_t + u(x)\rho_x = 0 \\ \rho(x<0, 0) = \frac{1}{2} \qquad \rho(x>0, 0) = -\frac{1}{2} \end{cases}$$

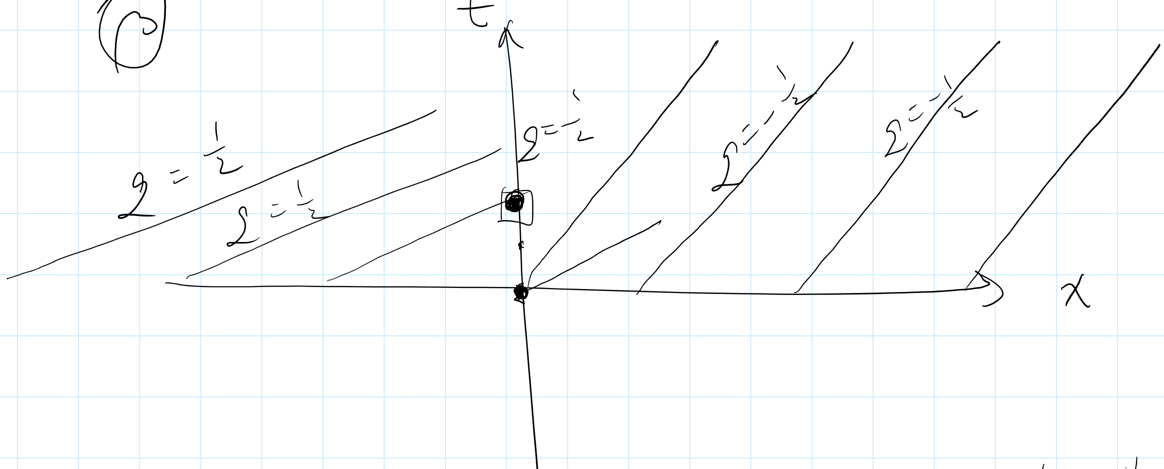
α $u(x) = 1$

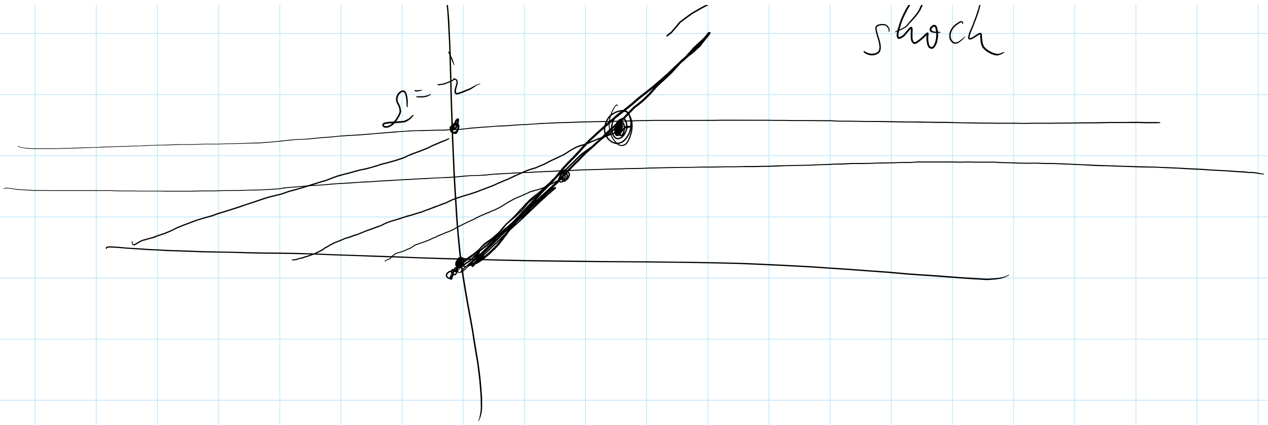
β $u(x<0) = 2 \qquad u(x>0) = 1$

α

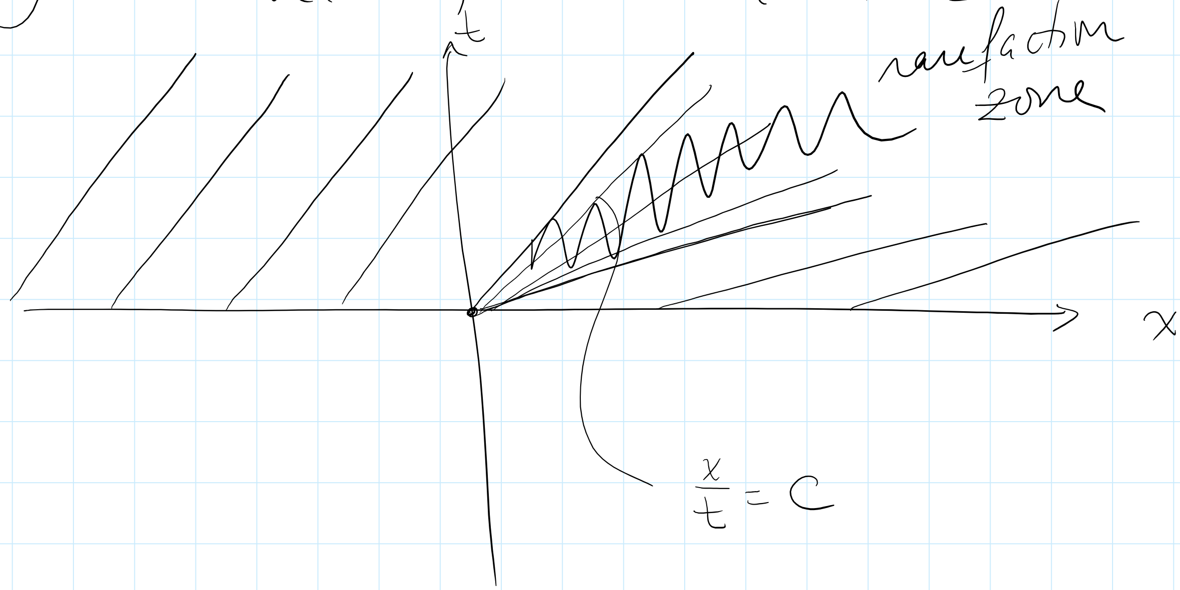


β





(4) $u(x<0) = 1 \qquad u(x>0) = 2$



$$\rho_t + u\rho_x = 0$$

Modified equations

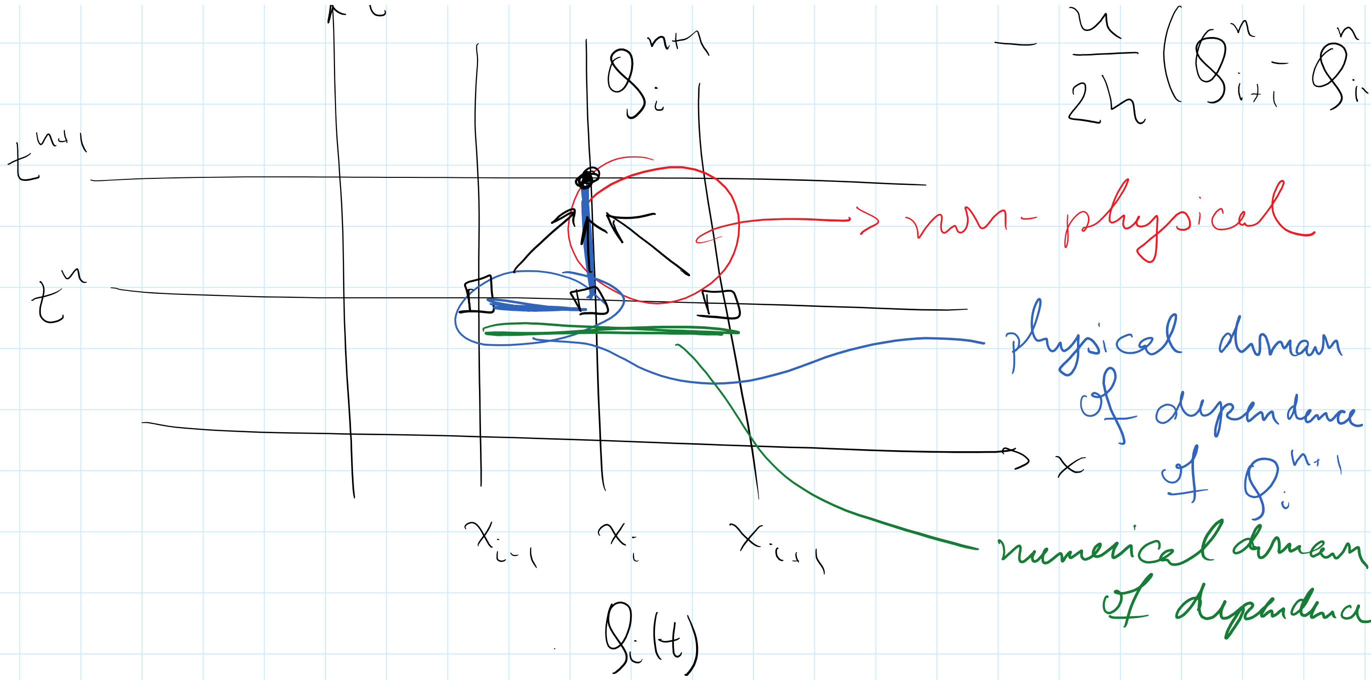
$$\rho_t + u\rho_x = 0 \quad ; \quad \rho_i \approx \rho(x_i, t)$$

$$\frac{d\rho_i}{dt} = -\frac{u}{2h}\left(\rho_{i+1} - \rho_{i-1}\right) \Rightarrow \frac{\rho_i^{n+1} - \rho_i^{n}}{k} = -\frac{u}{2h}\left(\rho_{i+1}^{n} - \rho_{i-1}^{n}\right)$$

t

ρ_i^{n+1}

$$-\frac{u}{2h}\left(\rho_{i+1}^n-\rho_{i-1}^n\right)$$



Courant, Friedrichs, Lewy

For a numerical scheme to converge to the solution it must be consistent & stable

Stability: domain of numerical dependence $\subseteq$ domain of physical dependence

<u>CFL condition</u> $\quad \nu$ = CFL number ≤ 1

$$\boxed{\nu=\frac{u\Delta t}{\Delta x}} \qquad \left(\nu=\frac{uk}{h}\right)$$

$$\rho_t+f(\rho)_x=0$$
$$\rho_t+f_\rho\,\rho_x=0$$

<u>Upwind scheme</u>.

$$\rho_t + u\rho_x = 0 \qquad u > 0$$

$$\frac{d\rho_i}{dt} = -\frac{u}{h}\left(\rho_i - \rho_{i-1}\right)$$

$$\rho_j^{n+1} = \rho_j^n - \nu\left(\rho_j^n - \rho_{j-1}^n\right) \qquad \rho_j^n = e^{i\xi j h} A^n$$

$$\left|\frac{A^{n+1}}{A^n}\right| < 1 \quad \text{(Von Neumann analysis)}$$

$$\rho_j^{n+1} = \rho_j^n - \nu\left(\rho_j^n - \rho_{j-1}^n\right)$$

$$(*)\ \rho(x_j, t^{n+1}) = \rho(x_j, t^n) - \nu\left(\rho(x_j, t^n) - \rho(x_{j-1}, t^n)\right)$$

Taylor series around $\rho(x_j, t^n)$

Is another PDE being approximated by $(*)$ to higher order than $\rho_t + u\rho_x = 0$

$$\rho_t + u\rho_x = A \cdot (\nu - 1)\rho_{xx}$$