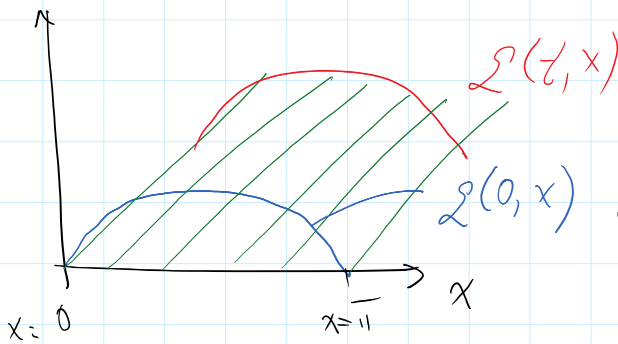
Modified equations

Monday, January 23, 2017 9:36 AM

Consider advection equation

$$\rho_t + u\rho_x = 0, \quad \rho(0,x) = \sin x$$

modeled numerically by upwind scheme



assume $\boxed{u>0}$

$$Q_j^n \cong \rho(t^n, x_j)$$

$$(1) \quad Q_j^{n+1} = Q_j^n - \nu\left(Q_j^n - Q_{j-1}^n\right)$$

$$\nu = \text{CFL number} = \frac{uk}{h}$$

Modified equation analysis. Take numerical scheme (1), replace num. approx Q by the function ρ

$$\rho(t^{n+1}, x_j) \cong \rho(t^n, x_j) - \nu\left(\rho(t^n, x_j) - \rho(t^n, x_{j-1})\right)$$

& Taylor-series expand around (t^n, x_j)

$\rho_t + u\rho_x = 0$

$\nu = \frac{uk}{x}$

$\rho_t = -\nu\rho_x$

$$\cancel{\rho} + k\rho_t + \frac{k^2}{2}\rho_{tt} + \frac{k^3}{6}\rho_{ttt} + \dots =$$

$$= \cancel{\rho} - \nu\left[\cancel{\rho} - \cancel{\rho} + h\rho_x - \frac{h^2}{2}\rho_{xx} + \frac{h^3}{6}\rho_{xxx} + \dots\right]$$

$\mathcal{O}(1): \mathcal{O}(k^0, h^0): \quad 0 = 0$ ✓

$\mathcal{O}(h): \mathcal{O}(k, h): \quad k\rho_t = -\nu h\rho_x$

$$O(1): O(k,h): \quad k\varrho_t = -\nu h \varrho_x$$

$$\nu = \frac{uk}{h} \Rightarrow$$

$$\varrho_t + u\varrho_x = 0 \quad \checkmark$$

$$O(2): O(k^2, h^2, kh)$$

$$k\varrho_t + \nu h \varrho_x = -\frac{k^2}{2}\varrho_{tt} + \frac{\nu h^2}{2}\varrho_{xx} + O(k^3, h^3, \dots)$$

$$\varrho_t + u\varrho_x = -\frac{k}{2}\varrho_{tt} + \frac{uh}{2}\varrho_{xx}$$

$$\varrho_t = -u\varrho_x \Leftrightarrow \partial_t \varrho = (-u\partial_x)\varrho$$

$$\varrho_{tt} = (\partial_t)^2\varrho = u^2\partial_x^2\varrho = u^2\varrho_{xx}$$

$$\varrho_t + u\varrho_x = \left(-\frac{k}{2}u^2 + \frac{uh}{2}\right)\varrho_{xx} =$$

$$= D\varrho_{xx} + O(k^2, \dots)$$

$$\varrho_t + u\varrho_x = O(k,h)$$

$$\varrho_t + u\varrho_{xx} = D\varrho_{xx} + O(k^2, \dots)$$

Artificial diffusivity

$$D = \frac{uh}{2}(1-\nu)$$

Observations

Observations

1) $\Delta = O(\text{base numerical scheme})$

2) $\Delta = 0$ for $\nu = 1$

Relevance to practical applications

1) Gas dynamics

Water Navier-Stokes

$$\begin{cases} \nabla \cdot \vec{u} = 0 \\ (\rho \vec{u})_t + \rho(\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \mu \nabla^2 \vec{u} \end{cases} \quad \text{(incompressible)}$$

μ = dynamic viscosity ;

ν = kinematic — " — $= \frac{\mu}{\rho} \approx O\left(10^{-6} \frac{m^2}{s}\right)$

$$\text{(NS)} \quad \vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \underline{\nu \nabla^2 \vec{u}}$$

ν small $\Rightarrow$ approximate (NS) eqs. by Euler equations

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p$$

Air : $\rho_t + \nabla(\rho \cdot \vec{u}) = 0$

$$Q = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix} \qquad Q_t + f(Q)_x + g(Q)_y + h(Q)_z = F(Q, Q_x)_x$$

$$f(Q) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ \rho u H \end{pmatrix} \quad \& \quad \text{cycl}$$

$$\begin{pmatrix} \rho u w \\ \rho u H \end{pmatrix}$$

$$F(\underline{q}) = \begin{pmatrix} \overset{0}{\mu(u_x - v_y)} \\ \vdots \\ \vdots \end{pmatrix}$$

Modified eq. for Lax-Wendroff scheme

Lax-Wendroff = Taylor series method + Centered FD approx. of spatial derivatives

$$q_t = -u q_x \qquad u \in \mathbb{R}$$

$$q(t+k, x) = q + k q_t + \frac{k^2}{2} q_{tt} + O(k^3)$$

$$= q - uk\, q_x + \frac{k^2 u^2}{2} q_{xx} \implies$$

$$Q_j^{n+1} = Q_j^n - \frac{\nu}{2}\left(Q_{j+1}^n - Q_{j-1}^n\right) + \frac{\nu^2}{2}\left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right)$$

$$(q_x)_j^n \approx \frac{Q_{j+1}^n - Q_{j-1}^n}{2h} \quad ; \quad (q_{xx})_j^n = \frac{Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n}{h^2}$$

Lax-W.

$$\cancel{q} + k q_t + \frac{k^2}{2} q_{tt} + \frac{k^3}{6} q_{ttt} + \ldots = \cancel{q} - \frac{\nu}{2}\left(q + h q_x + \cancel{\frac{h^2}{2} q_{xx}} + \frac{h^3}{6} q_{xxx} - q + h q_x - \cancel{\frac{h^2}{2} q_{xx}} + \frac{h^3}{6} q_{xxx} + \ldots\right)$$

$$+ \frac{\nu^2}{2}\left(q + h q_x + \frac{h^2}{2} q_{xx} + \cancel{\frac{h^3}{6} q_{xxx}}\right.$$

$$+\frac{\nu^2}{2}\Big(\rho + h\rho_x + \frac{h^2}{2}\rho_{xx} + \cancel{\frac{h^3}{6}\rho_{xxx}}$$

$$-2\rho$$

$$\rho - h\rho_x + \frac{h^2}{2}\rho_{xx} - \frac{h^3}{6}\rho_{xxx} + \dots\Big)$$

$O(0)$ (k^0, h^0): $\rho = \rho$ ✓

$O(1)$ $\rho_t = -u\rho_x$

$O(2)$ (k^2, h^2, kh): $\frac{k^2}{2}\rho_{tt} = \frac{\nu^2 h^2}{2}\rho_{xx}$ $\quad \rho_{tt} = u^2\rho_{xx}$ ✓

$O(3)$ $(k^3, h^3, kh^2, \dots)$: $\frac{k^3}{6}\rho_{ttt} = -\frac{\nu}{6}h^3\rho_{xxx}$

Artificial dispersivity of Lax-Wendroff

Reading assignment SIAM Rev. v1982

Group velocity in F.D.S.

L.N. Trefethen

$$\rho_t = -u\rho_x$$

$$\rho_j^n \doteq \rho(t^n, x_j);$$

$$t^n = nk; \quad x_j = jh$$

$$\rho(t+k) = \rho + \rho_t k + \rho_{tt}\frac{k^2}{2} + O(k^3)$$

$$\rho_j^{n+1} = \rho_j^n - \frac{\nu}{2}\left(\rho_{j+1}^n - \rho_{j-1}^n\right) +$$

$$\rho_j^{n+1} = \rho_j^n - \frac{\nu}{2}\left(\rho_{j+1}^n - \rho_{j-1}^n\right) + \frac{\nu^2}{2}\left(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n\right)$$

$$\rho_t = -u\rho_x + \underline{k u h^2 (1-\nu^2)}\, \rho_{xxx}$$

$$\rho(t+k) = \rho + k\rho_t + \frac{k^2}{2}\rho_{tt} + \frac{k^3}{6}\rho_{ttt} + O(k^4)$$

$$\rho_j^{n+1} = \rho_j^n - \frac{\nu}{2}\left(\rho_{j+1}^n - \rho_{j-1}^n\right) + \frac{\nu^2}{2}\left(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n\right) - \frac{\nu^3}{\;}\,\frac{\rho_{j+2}^n - \rho_{j+1}^n + \rho_{j-1}^n - \rho_{j-2}^n}{\;}$$

H.W. (have fun!)

Modified eq

$$\rho_t = -u\rho_x + u k h^3 (1-\nu) P_2(\nu)\, \underline{\rho_{xxxx}}$$

hyper-diffusive