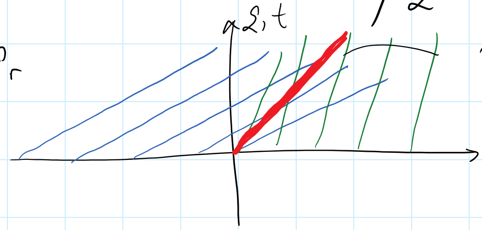
Burger's equation (simplest non-linear problem)

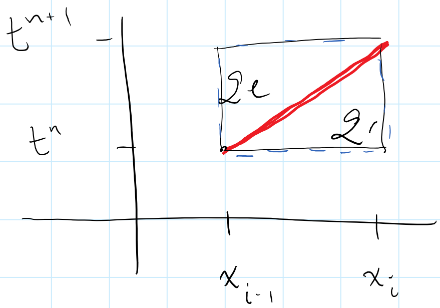
Riemann problem:
$$\begin{cases} \rho_t + \rho\rho_x = 0 \\ \rho(t=0,x) = \begin{cases} \rho_l & x<0 \\ \rho_r & x>0 \end{cases} \end{cases}$$

$\boxed{\text{Two cases:}}$ 1) $\rho_l > \rho_r$



$x = st$ s shock speed

Hugoniot relations give shock speed



$\rho_t + f(\rho)_x = 0$ differential formulation

weak formulation is obtained from
$$\int_{t^n}^{t^{n+1}}\int_{x_{i-1}}^{x_i} [\rho_t + f(\rho)_x]\,\varphi(t,x)\,dx\,dt = 0$$
by integration by parts that transfers ∂_t, ∂_x to $\varphi \in C^1$

Useful in consideration of (P. Lax) limit of viscous problem
$\rho_t + \rho\rho_x = \nu\rho_{xx}$ has solution $\rho_\nu(t,x)$

Ask: $\lim_{\nu\to 0} \rho_\nu(t,x)$; is it the solution of $\rho_t + \rho\rho_x = 0$.

Choose $\varphi = 1$
$$\int_{x_{i-1}}^{x_i} [\rho(t^{n+1},x) - \rho(t^n,x)]\,dx +$$
$$(1) \qquad \int_{t^n}^{t^{n+1}} [f(\rho(t,x_i)) - f(\rho(t,x_{i-1}))]\,dt = 0$$

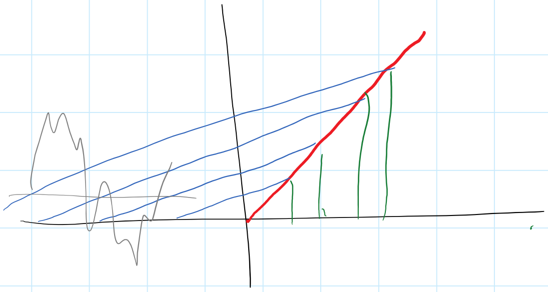
Apply Riemann problem initial condition to (1)

$n+1$ $\quad f(\rho_r) - f(\rho_l)$

Apply Riemann problem initial condition to (1)

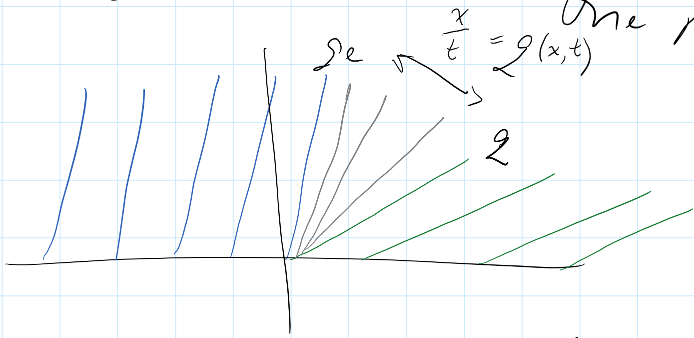
$$x_i - x_{i-1} = s(t^{n+1} - t^n)$$

$$\boxed{s = \frac{f(\rho_r) - f(\rho_l)}{\rho_r - \rho_l}}$$



Shocks "destroy" information that was "preserved" in propagation along characteristics

2) $\rho_l < \rho_r$



One possible solution: rarefaction fan, similarity solution $\rho(t,x)$ is replaced by $\rho_f\left(\frac{x}{t}\right) = \rho(t,x)$

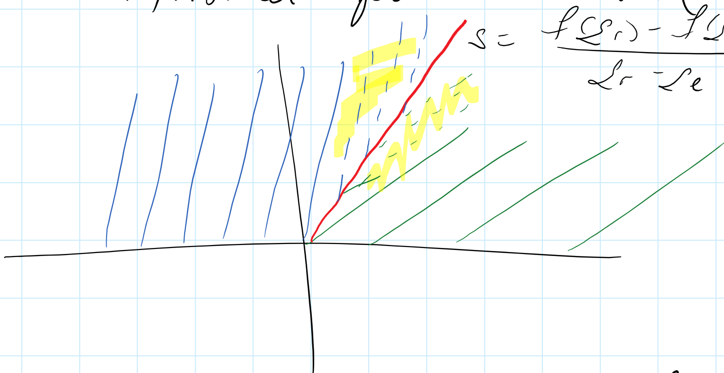
$$\rho_t + \rho\rho_x = \rho_f' \cdot \left(-\frac{x}{t^2}\right) + \rho_f \rho_f' \cdot \left(\frac{1}{t}\right) = 0 \Rightarrow$$

$$\frac{1}{t}\rho_f' \left[\rho_f - \frac{x}{t}\right] = 0$$ if $\frac{x}{t} = \rho_f \Rightarrow$ Burgers equation is satisfied

$$\rho_t + \rho\rho_x = 0$$

$$f(\rho) = \frac{\rho^2}{2}$$

Another possible solution



$$s = \frac{f(\rho_r) - f(\rho_l)}{\rho_r - \rho_l} = \frac{\rho_r^2 - \rho_l^2}{2(\rho_r - \rho_l)} = \frac{\rho_r + \rho_l}{2}$$

From point of view of inviscid Burgers $\rho_t + \rho\rho_x = 0$

__Both__ solutions are equally valid

Only the rarefaction solution is physically meaningful as:

a) the limit of solutions to $\varrho_t + \varrho\varrho_x = \nu\varrho_{xx}$ when $\nu \to 0$

&

b) entropy condition $(dS \geq 0)$

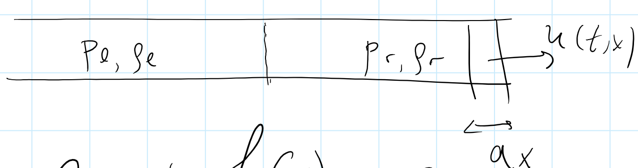
(Stat. phys. $S = -\int \psi(\omega) \underbrace{\log \psi(\omega)}_{\text{information}} \, d\omega$

$\psi(\omega)$ probability density function

(P. Lax 70's, classic of PDE analysis 50s-60s)

Euler equations of gas dynamics

$\boxed{1D}$



shock tube experiment

$$\varrho_t + f(\varrho)_x = 0$$

$$\varrho = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} = \begin{pmatrix} \rho \\ \ell \\ \varepsilon \end{pmatrix}; \quad f(\varrho) = \begin{pmatrix} \rho u \\ \rho u u + p \\ \rho H u + \rho \frac{p}{\rho} u \end{pmatrix} = \begin{pmatrix} \ell \\ \frac{\ell^2}{\rho} + p(\rho, \varepsilon) \\ \cdots \end{pmatrix}$$

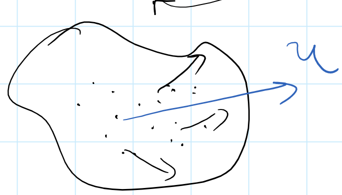
conservative variables

mechanical work

$f(\varrho)$ = physical flux = (resolved flux at scale of description) + (unresolved flux — " —)

$E = e + \frac{\rho u^2}{2}$ = total internal energy

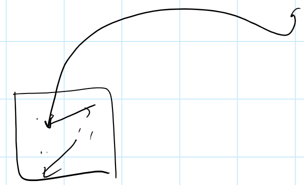
internal energy (unresolved "thermal" motion)



ρu^2

$$H = \text{total enthalpy} = h + \frac{\rho u^2}{2}$$

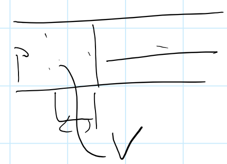
enthalpy



$$h = e + \frac{p}{\rho}$$

$\left(\frac{p}{\rho} \right.$ mechanical work

$$W = \int \vec{F} \cdot d\vec{x}$$



$\frac{p}{\rho} = p v$

v specific volume$\left. \right)$

(c.f. Landau & Lifschitz
Fluid Mechanics)

Solving the Riemann problem for Euler equations.

As a first step: solve the shallow water equations