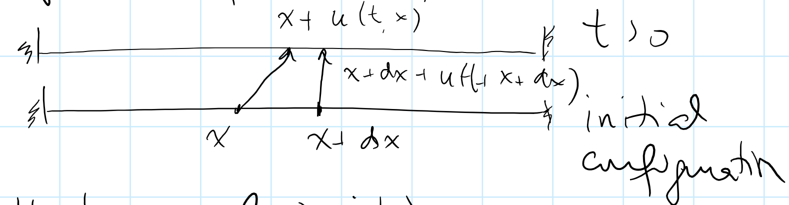
Elasticity

Wednesday, February 22, 2017 11:13 AM

Lagrangian reference frame



1D elasticity

Linear constitutive law (Hookean elasticity)

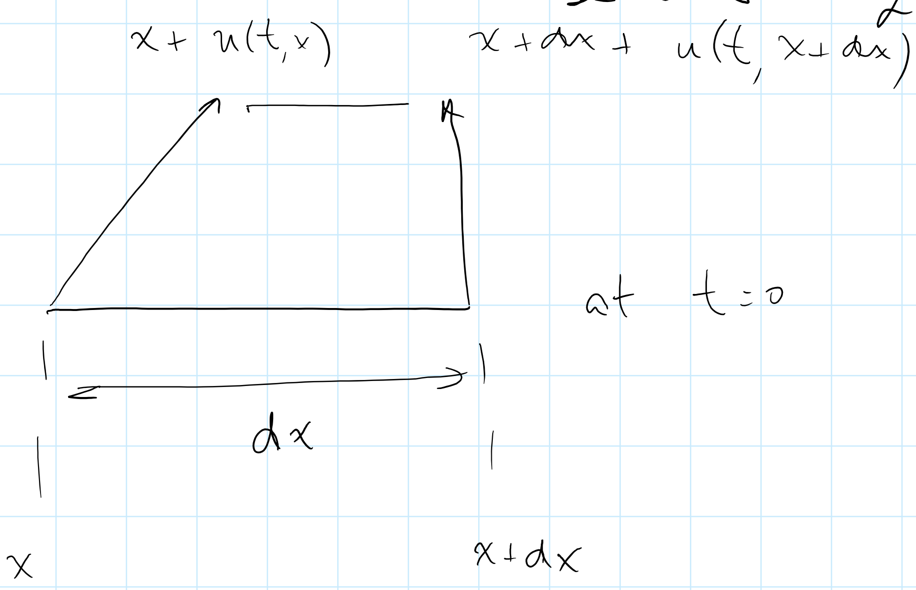
Kinematics

Dynamics

Constitutive relations

Initial configuration $\qquad u(t=0, x) = 0$

u = displacement from an initial configuration (assumed to be at static equilibrium)



at $t=0$

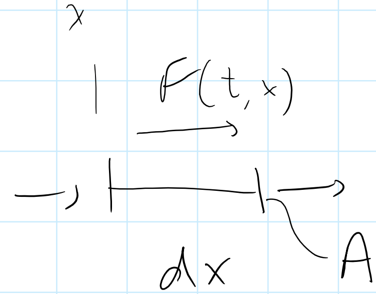
Kinematics

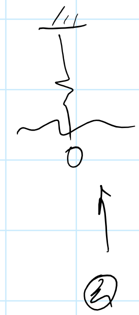
$$
\begin{aligned}
dl &= x + dx + u(t, x+dx) - (x + u(t,x)) \\
&= \underline{dx} + \underline{u(t, x+dx) - u(t,x)} \\
&= [1 + u_x]\, dx
\end{aligned}
$$

$= [1 + u_x]\, dx$

$v(t,x) = u_t$

$\frac{dl}{dx} = 1 + u_x$





Dynamics $\quad \lambda$ = linear density

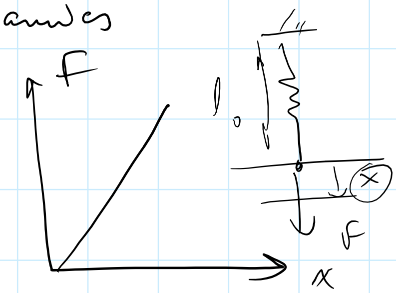
$$\underbrace{(\lambda\, dx)}_{\text{mass}}\, u_{tt}\left(t, x + \frac{dx}{2}\right) =$$

$$= \underbrace{F(t,x)\, dx}_{\text{body force}} + \left[\sigma(t, x+dx) - \sigma(t,x)\right] A$$

A = cross-sectional area, assumed to be constant

$$\lambda \underline{u_{tt}} = f + \underline{\sigma_x} A \quad ; \quad \lambda = \rho A$$

Constitutive laws: in general formulate closure between kinematics & dynamics



Hooke's law for a spring

Stress = force per unit area

Strain = relative displacement

$\sigma = E\, \varepsilon \qquad \sigma$ = normal stress $= \frac{F}{A}$

$$\varepsilon = \text{longitudinal strain} = \frac{\Delta l}{l_0} = \frac{\text{displacement}}{\text{initial length}}$$

$$\frac{F}{A} = E\,\frac{x}{l_0} \iff F = \frac{AE}{l_0}\,x \implies k = \frac{AE}{l_0}$$

$$\begin{cases} \sigma = E\,\varepsilon \\ \lambda u_{tt} = F + \sigma_x A \\ dl = (1+u_x)\,dx \implies \varepsilon = u_x = \dfrac{u_x dx}{dx} \\ \varepsilon_t = u_{xt} = u_{tx} = v_x \end{cases}$$

$$\lambda = \rho A \;; \quad F = f A \rho$$

$$\begin{cases} \rho v_t = \rho f + \sigma_x & \text{dynamics} \\ v_x = \varepsilon_t & \text{kinematics} \end{cases}$$

$$\sigma = E\,\varepsilon$$

$$\begin{matrix} (\text{i}) \\ (\text{ii}) \end{matrix} \begin{cases} \rho v_t - \sigma_x = \rho f \\ \frac{1}{E}\sigma_t - v_x = 0 \end{cases} \iff q_t + A q_x = \psi$$

$$q = \begin{pmatrix} v \\ \sigma \end{pmatrix} \;; \quad A = \begin{pmatrix} 0 & -E \\ -\frac{1}{\rho} & 0 \end{pmatrix} ; \; \psi = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$q = \begin{pmatrix} \dot{} \\ \sigma \end{pmatrix} ; \quad A = \begin{pmatrix} & \\ -\frac{1}{\rho} & 0 \end{pmatrix} ; \quad \psi = \begin{pmatrix} \\ f \end{pmatrix}$$

$$\det(A - \lambda I) = 0 \iff \lambda^2 - \frac{E}{\rho} = 0$$

$$\lambda_{1,2} = \sqrt{\frac{E}{\rho}} \mp$$ P-wave velocity in elastic
longitudinal velocity medium

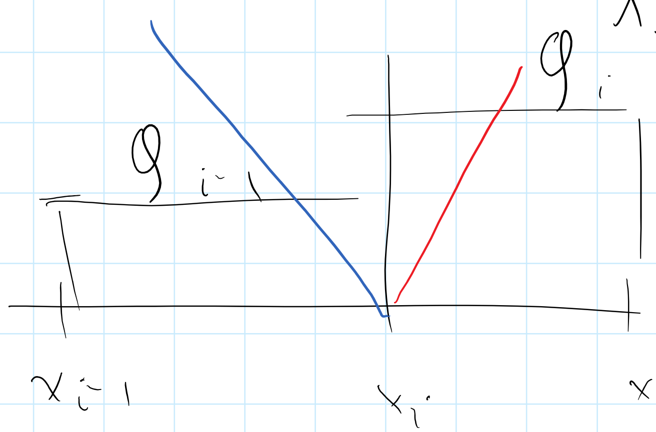
$[\lambda]$ = should be units of velocity

$$= \left[\frac{N}{m^2} \frac{m^3}{kg}\right]^{1/2} = \left[\frac{kg\, m}{s^2} \frac{m^3}{m^2\, kg}\right]^{1/2} = \frac{m}{s}$$

$$\begin{pmatrix} \sqrt{\frac{E}{\rho}} & -E \\ -\frac{1}{\rho} & \sqrt{\frac{E}{\rho}} \end{pmatrix} \Rightarrow \lambda_1 = \sqrt{\frac{E}{\rho}} ; \quad r_1 = \begin{pmatrix} 1 \\ \frac{c}{E} \end{pmatrix}$$

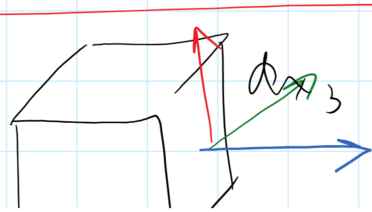
$$\sqrt{\frac{E}{\rho}} \cdot 1 - E \cdot r_{12} = 0$$

$$\lambda_2 = -\sqrt{\frac{E}{\rho}} , \quad r_1 = \begin{pmatrix} 1 \\ -\frac{c}{E} \end{pmatrix}$$

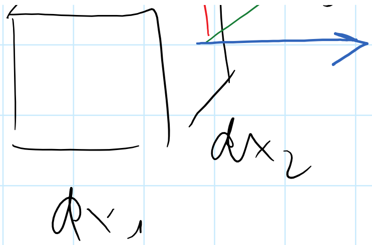


Riemann problem

Continuum Mechanics



$$\rho \dot{\vec{v}}\, dV = \rho f\, dV +$$

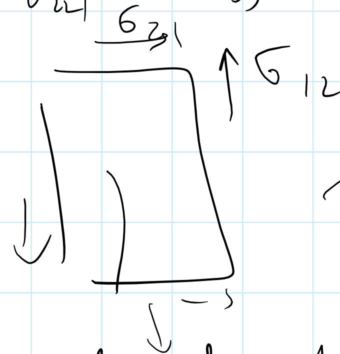


$$\rho \dot{\vec{v}}\, dV = \underline{\rho f\, dV} +$$

volume distributed forces "body" forces

$$\sigma = \text{stress tensor} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ \sigma_{21} & & \sigma_{33} \end{pmatrix}$$

$$\sigma = \sigma^T$$



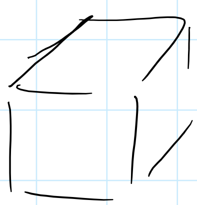
Landau Lifshitz

$$\rho\, \underline{dx_1\, dx_2 \cdot 1}\, a = \sigma_{12}\, dx_2 - \sigma_{21}\, dx_1$$

$$ma = \sum F$$

$$I \ddot{\theta} = \sum M$$

$$\sigma_{12} \cdot dx_2$$



Fluid

$$\sigma = -p I + \tau$$

$$p = -\frac{1}{3} \operatorname{Tr} \sigma$$

"isotropic component of normal stress"
a.k.a. pressure