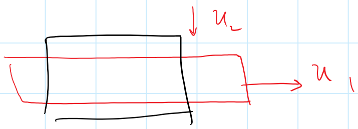
L17: General elasticity

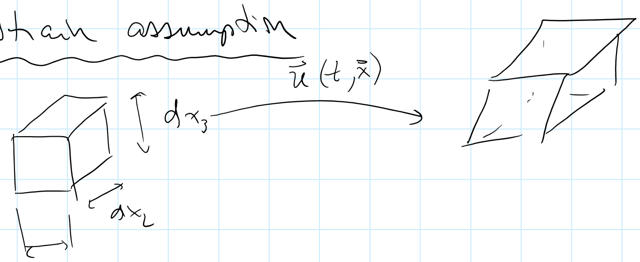
Friday, February 24, 2017 11:16 AM



Body in reference state

Poisson ratio ν
quantifies displacement ratio in orthogonal directions

Small strain assumption



Reference config
(often at equil. & 0-stress)

$\vec{u}(t,\vec{x})$ = displacements

σ = stress tensor
ϵ = strain tensor
} related by constitutive law

Kinematics $\quad v_i = u_{i,t}$ (Einstein differentiation & summation index conventions)

u_i vector $\quad \sigma_{ij}$ 2-rank tensor

$$\sigma_{ii} = \sum_{i=1}^{3} \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33};$$

summation convention

$$\mathrm{Tr}(\sigma) = \frac{1}{3}\sigma_{ii}$$

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}$$

Ex of index notation: Eulerian fl. dynamics

$$\rho_{,t} + (\rho v_i)_{,i} = 0$$

$$(\rho v_i)_{,t} + (\rho v_i v_j)_{,j} = -(p\delta_{ij})_{,j}$$

vector $\iff$ notation

$$\rho_t + \nabla\cdot(\rho\vec{v}) = 0$$

$$(\rho\vec{v})_t + \nabla\cdot(\rho\vec{v}\vec{v}) = -\nabla p$$

$\vec{v}\,\vec{v}$ = exterior product

δ_{ij} = Kronecker delta

ϵ_{ijk} = (completely anti-symmetric)

Levi-Civita tensor

$$\epsilon_{ijk} = \begin{cases} 1 & i,j,k \text{ is even perm of } 1,2,3 \\ -1 & \text{— " — odd perm} \\ 0 & \text{for any repeated index} \end{cases}$$

$$u_i = \epsilon_{ijk} v_j w_k \overset{\text{vector}}{\Longleftrightarrow} \vec{u} = \vec{v} \times \vec{w}$$

$$\vec{\omega} = \nabla \times \vec{u} \Rightarrow \epsilon_{ijk} u_{k,j} = \omega_i$$

Kinematics: $v_i(t,x) = u_{i,t}(t,x)$

$v_i(t,x)$ = Lagrangean coordinates, i.e., $v_i(t,x)$ is velocity of particle initially at position x in the reference configuration.

Strain tensor $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

(in vec. not.: $\epsilon = \frac{1}{2}(\nabla u + \nabla u^T)$)

<u>Dynamics</u>: $\rho u_{i,tt} = \rho v_{i,t} = f_i + \sigma_{ij,j}$

<u>Constitutive law</u>: in General:

vector form $\sigma = C : \epsilon$

C = 4-rank tensor

$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$

Hooke's law $\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}$

(λ, μ) — 1st, 2nd Lamé parameters

$[\lambda] = [\mu] = [E] = [G] = [p] = \frac{N}{m^2}$

$[\lambda] = [\mu] = [E] = [\sigma] = [p] = \dots m^2$

$$\lambda = \frac{\nu}{(1+\nu)(1-2\nu)} E \qquad \mu = \frac{1}{2(1+\nu)} E$$

ν = Poisson ratio

arises from sign convention

$$\frac{\partial \sigma_{11}}{\partial x_1} > 0$$

$$(*) \quad \begin{cases} \rho\, v_{i,t} = f_i + \sigma_{ij,j} \\ \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \\ \epsilon_{ij,t} = \frac{1}{2}(v_{i,j} + v_{j,i}) \\ \sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} \end{cases}$$

$$(**) \quad \sigma_{ij,t} = \lambda \delta_{ij} (v_{k,k}) + \mu (v_{i,j} + v_{j,i})$$

Velocity - Stress formulation of linear elas:

$$\begin{cases} v_{i,t} - \frac{1}{\rho} \sigma_{ij,j} = f_i \quad \left(\text{e.g.}, \ v_{1,t} - \frac{1}{\rho}\sigma_{11,1} - \frac{1}{\rho}\sigma_{12,2} - \frac{1}{\rho}\sigma_{13,3} = f_1\right) \\ \sigma_{ij,t} - \lambda \delta_{ij} v_{k,k} - \mu (v_{i,j} + v_{j,i}) = 0 \end{cases}$$

$$q_t + f(q)_x + g(q)_y + h(q)_z = 0$$

$$q = (\sigma_{11} \ \sigma_{22} \ \sigma_{33} \ \sigma_{12} \ \sigma_{23} \ \sigma_{31} \ v_1 \ v_2 \ v_3)$$

$$q_t + A q_x + B q_y + C q_z = 0$$

$-\lambda-2\mu$

$$A = \begin{array}{c|c} \begin{pmatrix} & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ -1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \\ v_1 \\ v_2 \\ v_3 \end{matrix} \\ \hline \begin{matrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{31} & v_1 & v_2 & v_3 \end{matrix} & \end{array}$$

$$\sigma_{ij,t} - \lambda \delta_{ij} v_{k,k} - \mu (v_{i,j} + v_{j,i}) = 0$$

$$\sigma_{11,t} - \lambda\,(v_{1,1} + v_{2,2} + v_{3,3}) - 2\mu v_{1,1} = 0$$

$$\sigma_{11,t} - (\lambda + 2\mu)\, v_{1,1} - \lambda v_{2,2} - \lambda v_{3,3} = 0$$

$$q_t + A q_x + B q_y + C q_z = 0$$

$$\mathcal{J} = (A q,\ B q,\ C q)$$

$A r = \lambda r \Rightarrow$ eigensystem $\Rightarrow$

1) P-waves (longitudinal waves) (forward & backward)
2) S-waves (shear waves) (—"—, along 2 directions)

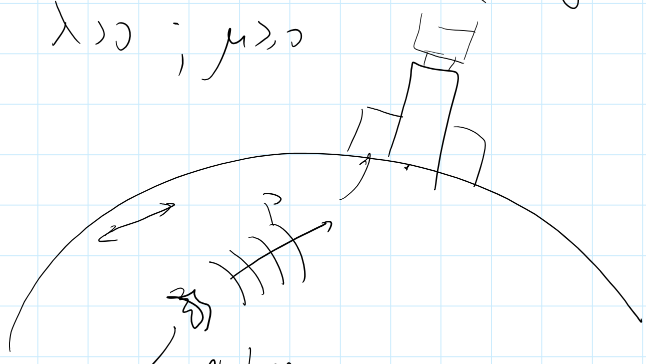
6 propagating waves

3 non-propagating waves

$$c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}} > c_s = \sqrt{\frac{\mu}{\rho}}$$

Dimensional homogeneity $\left(\frac{N}{m^2}\frac{m^3}{kg}\right)^{\frac{1}{2}} = \left(\frac{kg\ m}{s^2}\frac{m^3}{m^2\ kg}\right)^{\frac{1}{2}} = \frac{m}{s}$ ✓

$\lambda > 0$; $\mu > 0$



$$q_t + \vec{\mathcal{J}} \cdot \nabla q = 0$$

$$\vec{\mathcal{J}} = A\vec{i} + B\vec{j} + C\vec{k}$$

Choose a projection direction $\vec{n} = (n_x, n_y, n_z)$

$$q_t + (\vec{\mathcal{J}} \cdot \vec{n})\, q_n = 0$$ projection of conservation.

flow (elasticity)
along direction n.