Variational principles

Monday, March 20, 2017 11:18 AM

overview Math

Consider the problem of minimizing a functional.

def A functional is a mapping from a linear space to scalars

Ex: a) $f \in C[a,b] \longmapsto \int_a^b f(x)\,dx$

b) $f, g \in C[a,b] \longmapsto \int_a^b w(x) f(x) g(x)\,dx$

R. Moore: "Computational Functional Analysis" (< 300pp)

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overview from physics.

All physics is derived from minimization of action

$$\mathcal{S}(q, \dot q, t) = \int_{\text{trajectory}} \mathcal{L}(q, \dot q, t)\,dt$$

$q(t), \dot q(t) : \mathbb{R}_+ \longmapsto \mathbb{R}^m$ $\quad m =$ no. of Degrees of Freedom of system

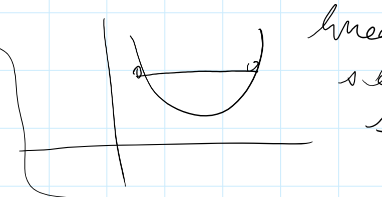
Action minimization is w.r.t. $q(t), \dot q(t)$

$\delta \mathcal{S} =$ change in action due to $\delta q, \delta \dot q$

$$\delta \mathcal{S} = \int \mathcal{L}(q + \delta q, \dot q + \delta \dot q, t)\,dt - \int \mathcal{L}(q, \dot q, t)\,dt$$

$$= \int \left(\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \dot q} \delta \dot q \right) dt = 0 \quad \text{(Stationarity condition)}$$

Stationarity corresponds to an extremum if $\mathcal{S}$ is convex



linear comb. stay within space

$$\delta \dot q = \delta \frac{dq}{dt} = \frac{d}{dt} \delta q$$

↳ assume commutativity

([illegible])

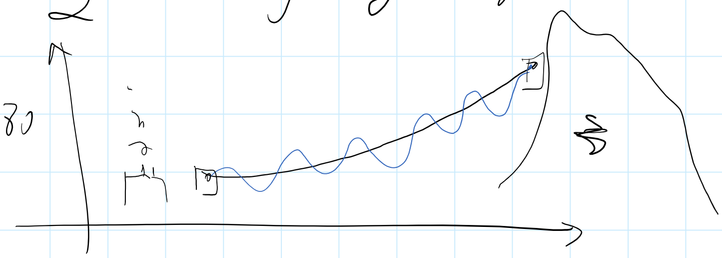
↳ assume commutativity

$$\delta \mathcal{S} = 0 = \int \left(\frac{\partial \mathcal{L}}{\partial q}\,\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\,\frac{d}{dt}\delta q\right) dt$$

$$= \int \left[\frac{\partial \mathcal{L}}{\partial q}\,\delta q - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\delta q\right] dt + \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\,\delta q\right]^{\text{endpoints of trajectory}}}_{0}$$

δq at trajectory endpoints $= 0$

Ex: ~1780



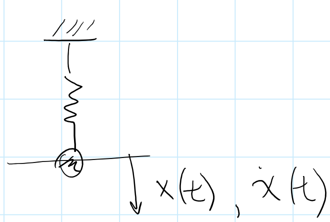
~1940 Feynman Path integration

$$\delta \mathcal{S} = 0 = \int \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right]\delta q \; dt$$

Stationarity w.r.t $\forall\, \delta q \Rightarrow \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0$

↳ Euler Variational Equations

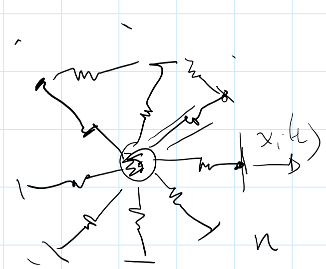
Ex:



$$\mathcal{L} = \text{potential energy} + \text{kinetic energy}$$

$$= -\frac{kx^2}{2} + \frac{m\dot{x}^2}{2}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow m\ddot{x} + kx = 0$$



$x_i(t)$, $\dot{x}_i(t)$

$$\mathcal{L} = \sum_{i=1}^{n}\left(-\frac{k_i x_i^2}{2} + \frac{m_i \dot{x}_i^2}{2}\right)$$

$u(x,y)$, $g(x,y)$

$\mathcal{L}(u, u_x, u_y)$

$$\mathcal{L} = \frac{1}{2} g u^2 - \frac{1}{2}\left(u_x^2 + u_y^2\right) \qquad (1)$$

Euler Variational Eq.: $\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial u_y}\right) - \frac{\partial \mathcal{L}}{\partial u} = 0 \quad (2)$

$$\delta S = \int\left(\frac{\partial \mathcal{L}}{\partial u}\delta u + \frac{\partial \mathcal{L}}{\partial u_x}\delta u_x + \frac{\partial \mathcal{L}}{\partial u_y}\delta u_y\right) dx\, dy - \int \mathcal{L}(u, u_x, u_y)\, dx\, dy$$

(1) (2): $-(u_{xx} + u_{yy}) = g u$

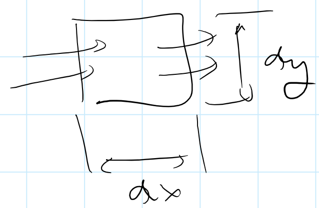
$$-\nabla^2 u = g u \qquad \text{(Helm. Eq.)}$$

Relevance for F.E.M.

Problems with a variational principle can apply var. principle to obtain a numerical formulation

Rayleigh-Ritz methods

Heat equation $\rho(t,x,y)$



$$\rho_t = \sigma(t,x,y,\rho) - \nabla \cdot \vec{f}(\rho)$$

$\sigma = 0$; $\vec{f}(\rho) = 0 + 0 \cdot \rho + \alpha \rho_x + \beta \rho_y$

constant term

leading order behavior

$\Rightarrow$ Fick's law mass diff

Fourier's law thermal diff "

Stokes law momentum diff