Ritz, Galerkin methods, FreeFEM

Wednesday, March 22, 2017 11:14 AM

$$\mathcal{I}(u, u_x, u_y) = \int_\Omega \mathcal{L}(u, u_x, u_y)\, dx\, dy$$

$\Omega \subset \mathbb{R}^2$ $\qquad \mathcal{L}(u, u_x, u_y) = \frac{1}{2}(u_x^2 + u_y^2) + g u$

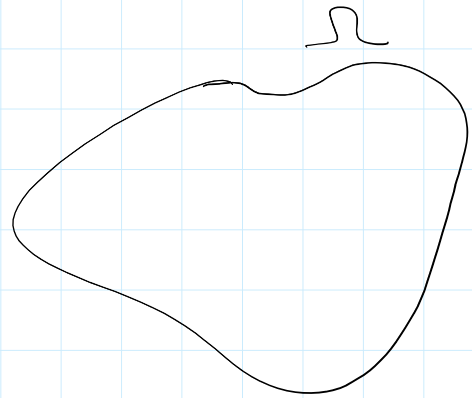
Euler equations giving $u(x,y)$ that solves $\min\limits_{u} \mathcal{I}$ $\quad\Big|\quad u_x = \partial_x u$, $u_y = \partial_y u$

are

$$\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial u_y}\right) - \frac{\partial \mathcal{L}}{\partial u} = 0 \Rightarrow$$

$$\begin{cases} u_{xx} + u_{yy} = g & \text{in } \Omega \\ u = f \text{ Dirichlet} & \text{on } \partial\Omega \\ \text{or } u_n = f \text{ Neumann} & \\ \text{or } u + k u_n = f \text{ Robin} & \end{cases}$$

Discretization spaces



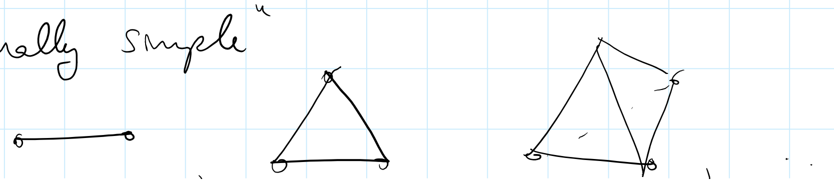
1) Discretization of geometry

$\tilde{\Omega}$ approximation of Ω

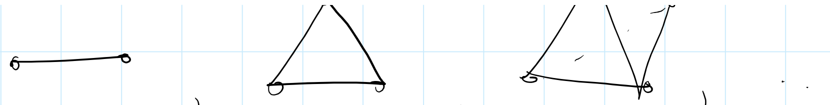
$\tilde{\Omega} \subseteq \Omega \qquad \tilde{\Omega} = \bigcup\limits_{i=1}^{N} \Omega_i$

Ω_i "computationally simple"

a) Simplicia



a) Simplices

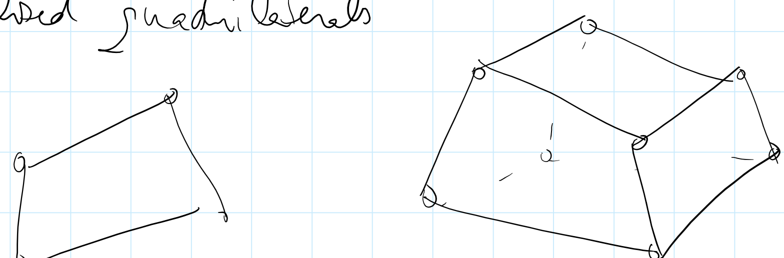


1-D, 2-D, 3-D, . . .

$$L = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix}, \quad A = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$V = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

b) Generalised "quadrilaterals"



2D 3D

---

2) Discretization in approximation space

Discretization of function space into an approx sp.

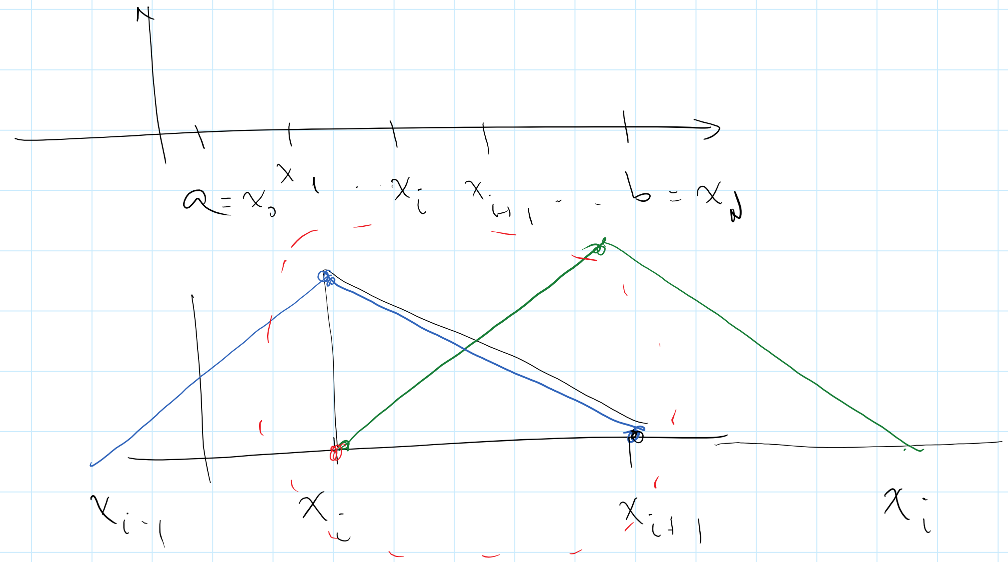
B.V.P. $\begin{cases} \nabla^2 \varphi = g \text{ in } \Omega & \varphi \in C^2(\Omega) \\ \varphi = f \text{ on } \partial\Omega \end{cases}$

$$\tilde{\varphi}(x,y) = \sum_{k=1}^{M} \varphi_k \underline{N_k(x,y)} \cong \varphi(x,y)$$

N_k "form functions" in F.E.M. literature

basis functions

Ex: 1D)



$\Omega = [a,b]$

$$\begin{cases} \varrho_{xx} = g \\ \varrho(a) = \alpha \\ \varrho(b) = \beta \end{cases}$$

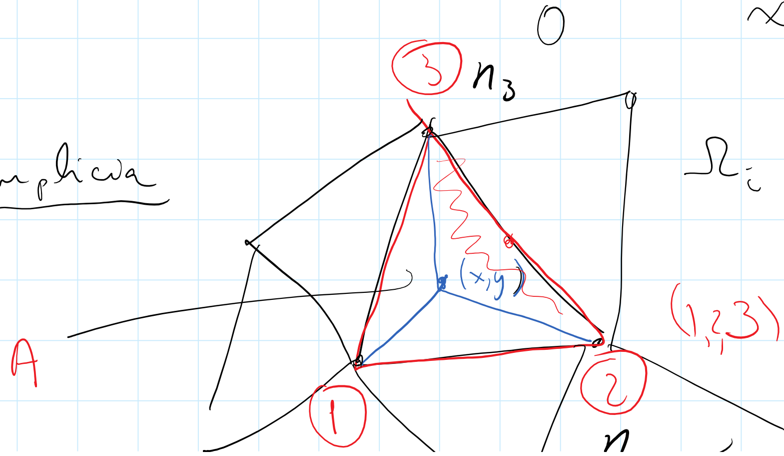
$\varrho_i \cong \varrho(x_i)$ Nodal values

$$\tilde{\varrho}(x) = \sum_{i=0}^{N} \varrho_i N_i(x) \qquad a \le x \le b$$

$\varrho_0 = \alpha$; $\varrho_N = \beta$

$$N_i(x) = \begin{cases} 0 & x \le x_{i-1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} < x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x_i < x \le x_{i+1} \\ 0 & x_{i+1} < x \end{cases}$$

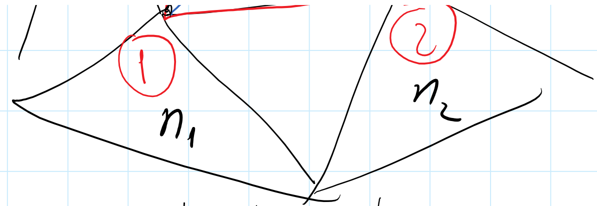
2D) Simplices



Ω_i = i^{th} simplicial element

(1,2,3) = local node numbering

4



numbering

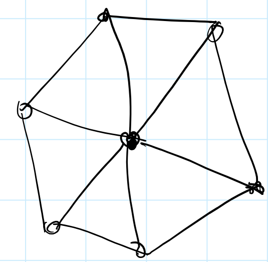
$$N_1(x,y) = \frac{1}{2A}\begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix} \; ; \quad N_1(x_1,y_1) = 1$$

$$N_1(x,y) = 0 \quad (x,y) \in \overline{2,3}$$

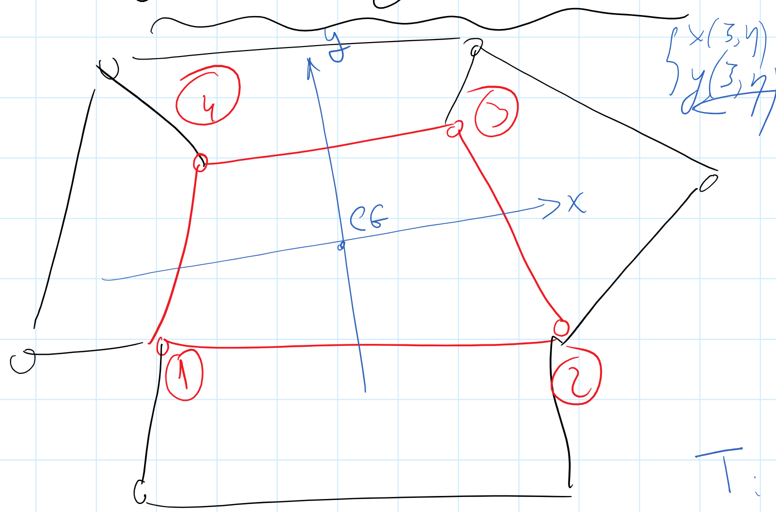
$$N_2(x,y) = \frac{1}{2A}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix}$$

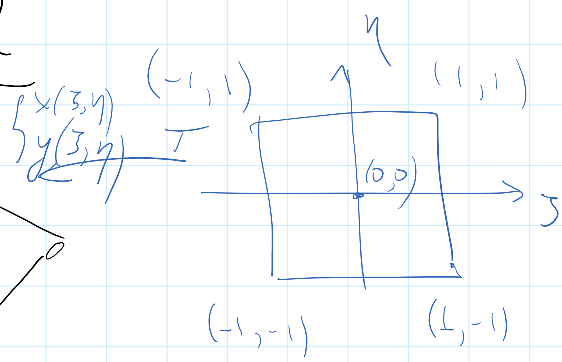
$$N_3(x,y) = \frac{1}{2A}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix}$$

$$\tilde{\varphi}(x,y) = \sum \varphi_i N_i(x,y)$$



Generalized quadrilateral





$$T : \begin{cases} x(\xi,\eta) = \sum_{k=1}^{4} X_k N_k(\xi,\eta) \\ y(\xi,\eta) = \sum_{k=1}^{4} Y_k N_k(\xi,\eta) \end{cases}$$

$$N_{1,2,3,4}(\xi,\eta) = \frac{1}{4}(1 \pm \xi)(1 \pm \eta)$$

$$\tilde{\varphi}_e(\xi,\eta) = \sum_{k=1}^{4} \varphi_k N_k(\xi,\eta)$$

$$\tilde{\varphi}(x,y) = \tilde{\varphi}_e \circ T^{-1}(x,y)$$

Rayleigh-Ritz formulation

$$\begin{cases} \varphi_{xx} + \varphi_{yy} = g & \text{in } \Omega \\ \varphi = f & \text{on } \partial\Omega \end{cases} \iff \min_{\varphi} \int_{\Omega} \mathcal{L}(\varphi_x, \varphi_y, \varphi)\, dx\, dy$$

$$\mathcal{L}(\varphi_x, \varphi_y, \varphi) = \frac{1}{2}\left(\varphi_x^2 + \varphi_y^2\right) + g\varphi$$

Introduce an approximation $\tilde{\varphi}(x,y) = \sum_k \varphi_k N_k(x,y)$

$$\min_{\{\varphi_k\}} \int \mathcal{L}(\varphi_x, \varphi_y, \varphi)\, dx\, dy \Rightarrow$$

$$\frac{\partial}{\partial \varphi_k} \int \mathcal{L}(\tilde{\varphi}_x, \tilde{\varphi}_y, \tilde{\varphi})\, dx\, dy = 0 \qquad k = 1, \dots, M$$

M no. of nodal values

$$\frac{\partial}{\partial \varphi_k} \int\left\{ \frac{1}{2}\left[\left(\sum_{j=1}^{M} \varphi_j \frac{\partial N_j}{\partial x}\right)^2 + \left(\sum_{j=1}^{M} \varphi_j \frac{\partial N_j}{\partial y}\right)^2\right] + g(x,y) \sum_{j=1}^{M} \varphi_j N_j(x,y) \right\} dx\, dy = 0$$

$$\int \left[\left(\sum_{j=1}^{M} \varphi_j \frac{\partial N_j}{\partial x}\right)\frac{\partial N_k}{\partial x} + \left(\sum_{j=1}^{M} \varphi_j \frac{\partial N_j}{\partial y}\right)\frac{\partial N_k}{\partial y} + g(x,y) N_k(x,y) \right] dx\, dy = 0$$

$$+ \; g(x,y)\, N_k(x,y) \Big]\, dx\, dy = 0$$

$$\sum_{j=1}^{M} \left(\int \left(\frac{\partial N_j}{\partial x} \frac{\partial N_k}{\partial x} + \frac{\partial N_j}{\partial y} \frac{\partial N_k}{\partial y} \right) dx\, dy \right) \phi_j =$$

$$= \int g(x,y)\, N_k(x,y)\, dx\, dy$$

$$b_k = \int g(x,y)\, N_k(x,y)\, dx\, dy$$

$$A_{kj} = \underline{\int (\nabla N_j) \cdot (\nabla N_k)}\, dx\, dy$$

$A \phi = b$ a linear system $\Rightarrow$

$A =$