Rayleigh-Ritz

Overview

$$(1)\quad \begin{cases} Lu = 0 & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega \end{cases}$$

L (differential) operator

B boundary operator

if L results from variational principle of action minimization

$\Rightarrow$ (1) is equivalent

$$\min_u \mathcal{J}(u, u', u'', \dots)$$

Galerkin Method

$$(2)\quad \begin{cases} Lu = 0 & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega \end{cases} \qquad (B = 0)$$

Finite element discretization

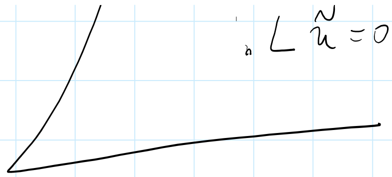
1) $\{\Omega_i\}_{i \in N}$ $\qquad \bigcup_{i=1}^{N} \Omega_i \cong \Omega$

2) $u(x) = \sum_{i=1}^{M} U_i N_i(x)$

Galerkin: Solution to (2) is approximated by the projection onto $\operatorname{span}\{N_1, \dots, N_M\}$

$Lu = 0$

$L\tilde{u} = 0$



$L\tilde{u} = 0$

$$Lu = 0 \iff$$

$$(3) \quad \langle L\tilde{u}, N_k \rangle = 0 \qquad k = 1, \dots, M$$

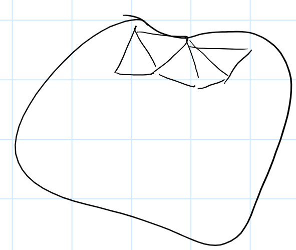
Example: $\langle f, g \rangle = \int_\Omega f(x)\, g(x)\, dx$

$$\int_\Omega \left[L\left(\sum_{i=1}^{M} U_i N_i(x) \right) \right] N_k(x)\, dx = 0$$

$$\iff \sum_{i=1}^{M} U_i \int_\Omega \left(L N_i(x) \right) N_k(x)\, dx = 0$$

$$AU = 0$$

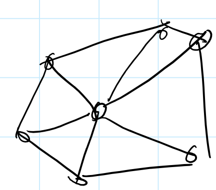
Ex: $\begin{cases} u_{xx} + u_{yy} = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$



$$\tilde{u}(x,y) = \sum_{i=1}^{N} U_i N_i(x,y) \qquad N = \text{no. of elements}$$

$u(x_i, y_i) = n_i U_i$ $\qquad$ n_i: connectivity = no. of elements that have (x_i, y_i) as a vertex

$$\langle L\tilde{u}, N_k \rangle =$$



$$\int_\Omega \nabla^2 \tilde{u}\, N_k(x)\, dx =$$

$$= -\int_{\Omega} (\nabla \tilde{u} \cdot \nabla N_k(x))\, dx + b(x) = \int f \, N_k(x)\, dx \Rightarrow$$

$$-\sum_{i=1}^{N} U_i \int \nabla N_i(x) \cdot \nabla N_k(x)\, dx = -b(x) + \int f \, N_k(x)\, dx$$

$$\Rightarrow \quad A \cdot U = y$$