Weak forms

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Goal: Solve $Au = f$ (1) Strong form

$u \in V$ V vector space, e.g.

- $\mathbb{R}^n$
- $C^m(\Omega)$
- $H^{m,p}(\Omega)$

A is an operator $A: V \to W$

Ex: 1) $A = \frac{d^2}{dx^2}$; $V = C^\infty$, $W = C^\infty$

2) $A = \frac{d^2}{dx^2}$; $V = C^m$; $W = C^{m-2}$

Motivation $\begin{cases} \rho_t + \rho\rho_x = 0 \\ \rho(x,0) = \begin{cases} 2 \\ 1 \end{cases} \end{cases}$; $\begin{cases} \rho_t + \rho\rho_x = 0 & \text{in } (0, 4\pi) \\ \rho(x,0) = \sin x \end{cases}$

Strategy: introduce a weak form

Weak form is inspired by variational considerations of least action

$$S(q(t), \dot{q}(t)) = \int_{t_0}^{t_1} \mathcal{L}(q, \dot{q})\, dt \iff \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$

$\min_{(q, \dot{q})} S$

$$\delta S = \int_{t_0}^{t_1} \left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q}\right) \delta q \, dt$$

(weak form)

$Aq = 0$

$A: V \to W$

(strong form)

$A : V \to W \qquad Au = f \qquad$ (Strong form)

Construct weak form $\qquad \langle Au, v\rangle = \langle f, v\rangle$

$$[Au](v) : V \to \mathbb{R}$$

Ex

Linear Systems	Poisson Eq
$A : \mathbb{R}^m \to \mathbb{R}^m \; ; \; u, f \in \mathbb{R}^m$	
$Au = f \qquad (1.1)$	$-\nabla^2 u = f$
$a(u,v) = v^T A u$	$a(u,v) = \int_\Omega \nabla u \, \nabla v \, d\Omega$
Choose test functions v s.t	$a(u,v) = \langle f, v\rangle$
if $\quad a(u,v) = \langle f, v\rangle$	$\iint_\Omega (u_x v_x + u_y v_y)\, dx\, dy =$
$\Rightarrow$ (1.1) is satisfied	$= \int_\Omega f \cdot v \; dx\, dy$
m finite $\Rightarrow v \in \{e_1, \dots, e_m\}$	
$e_j^T A u = e_j^T v$	$u = \sum u_i N_i(x,y)$
$u = \sum_{i=1}^{m} u_i e_i$	