Lax-Milgram theorems

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Framework for solving a PDE

By variational principle:

(1) $Au = f$

$A : V \to W$

A some operator

$a(u, v)$ associated bilinear form

Solution to (1) $\Leftrightarrow$ solution of $a(u,v) = \langle f, v \rangle$

by variational calculus

$A : \mathbb{R}^n \to \mathbb{R}^n$ linear operator

$$a(u, v) = \langle Au, v \rangle$$

$$u = u_i e_i \; ; \quad v = v_j e_j$$

$$\underline{\langle Au, v \rangle} = \langle f, v \rangle \Leftrightarrow$$

$$Au = f \qquad u, f \text{ vectors in } \mathbb{R}^n$$

Ex: $Au = f \qquad a(u,v) = \langle Au, v \rangle = u^T A^T v$

<u>Ex:</u> (1) $-\nabla^2 u = f \quad \Rightarrow \quad a(u,v) = \langle \nabla u, \nabla v \rangle$

(2) $a(u,v) = \langle f, v \rangle \quad \forall v \in V$

(1) $\Leftrightarrow$ (2)

Classical PDE theory 1920's, (Kellog book)

$$(BVP) \begin{cases} -\nabla^2 u = f \text{ in } \Omega \subset \mathbb{R}^3 \\ u\big|_{\partial\Omega} = g \end{cases}$$

Harmonic representation theorem

Harmonic representation theorem

General form

$$u_p = \frac{1}{4\pi}\int_{\Omega} \frac{f(Q)}{r_{PQ}}\, dr_Q + \frac{1}{4\pi}\int_{\partial\Omega} \frac{d}{dn}\left(\frac{1}{r_{PQ}}\right) g_1(r_Q)\, dr_Q + \frac{1}{4\pi}\int_{\partial\Omega} \frac{g_2(r_Q)}{r_{PQ}}\, dr_Q$$

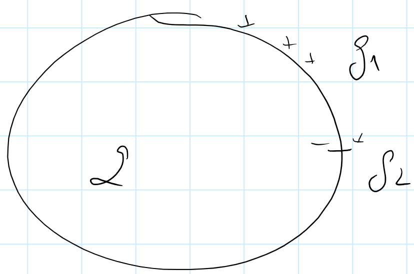
double layer potential

Volume potential

single layer potential

Ex: Electrostatics

$$-\nabla^2 V = \rho$$



Dirichlet B.C.

specify g_1 ; g_2 results

Neumann B.C.

specify g_2 ; g_1 results

Robin B.C.

specify linear combination

Summer Beach Reading List

- Morse & Feshbach
- Kellog "Potential theory"
- Courant & Hilbert "Meth. of Math. Phys."
- Schwartz "Theory of Distributions"

$Au = f \qquad A \in \mathbb{R}^{m\times m}$; $u, f \in \mathbb{R}^{n}$

Lax - Milgram Theorem

$$Au = f \ \underset{\text{calculus}}{\overset{\text{var.}}{\Longleftrightarrow}}\ a(u,v) = \langle f, v\rangle \quad (2)$$

$$Au = f \quad \underset{\text{calculus}}{\overset{\text{var.}}{\Longleftrightarrow}} \quad a(u,v) = \langle f, v \rangle \quad (2)$$

(strong form) weak form

(2) has a solution for $\forall f$ if

α) a is bounded $\quad |a(u,v)| \leq C \|u\| \|v\|$

C finite constant

β) a is coercive ; $\exists\, c > 0$ s.t.

$$a(u,u) \geq c \|u\|^2$$

Ex: Linear system $Au = f$

$$a(u,u) = \langle Au, u \rangle = u^T A^T u \geq \underline{c\, u^T u}$$

Lax-Milgram generalizations

Lax-Milgram $\quad a : H \times H \to \mathbb{R} \quad$ H Hilbert

Babuška-Lax-Milgram $\quad a : U \times V \to \mathbb{R} \quad$ U, V Hilbert

Lions-Lax-Milgram $\quad a : H \times V \to \mathbb{R} \quad$ H Hilbert, V normed

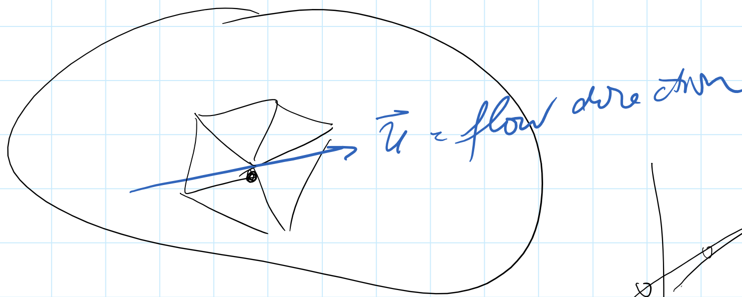
Lions: <u>Theory for PDEs & FEMs</u> (12 + 14 volumes)

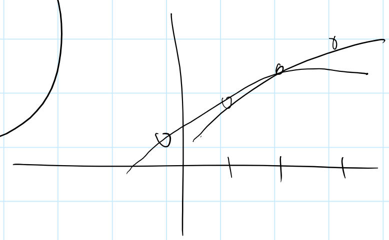
Textbook by Beatrice Rivière FEM Th. & Applications (~300p).

<u>HW3</u> = HW1 with FEM replacing FVM

FEM have difficulties with upwinding

(most common approach SUPG = stream-wise upwind Petrov-Galerkin)





Incompressible flow : (Navier-Stokes eqs.)

$$\begin{cases} \vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} = -\nabla p + \nu \nabla^2 \vec{u} \\ \nabla\cdot\vec{u} = 0 \end{cases}$$

is solved by operator splitting (Projection method Chorin 1969)

J. Computational Physics

Operator splitting stages

1) Advection

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} = -\nabla \chi$$

χ an approximation of p

$\vec{u}^n \to \vec{u}^*$

2) Diffusion

$$\vec{u}_t = \nu \nabla^2 \vec{u}$$

$\vec{u}^* \to \vec{u}^{**}$

3) Projection $\vec{u}^{n+1} = \vec{u}^{**} + \nabla\phi$

$$\nabla\cdot\vec{u}^{n+1} = 0 \Rightarrow$$

$\nabla^2 \phi = -\nabla \cdot \vec{u}^{**}$ (enforces $\nabla \cdot \vec{u} = 0$)