Outline

- Nonlinear hyperbolic systems
- Shallow water equations
- Shock waves and Hugoniot loci
- Integral curves in phase plane
- Compression and rarefaction

Notes:

Shallow water equations

$h(x,t) =$ depth
$u(x,t) =$ velocity (depth averaged, varies only with x)

Conservation of mass and momentum hu gives system of two equations.

mass flux $= hu$,
momentum flux $= (hu)u + p$ where $p =$ hydrostatic pressure

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{gh}.$$

Notes:

Shallow water equations

$$h_t + (hu)_x = 0 \implies h_t + \mu_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0 \implies \mu_t + \phi(h,\mu)_x = 0$$

where $\mu = hu$ and $\phi = hu^2 + \frac{1}{2}gh^2 = \mu^2/h + \frac{1}{2}gh^2$.

Jacobian matrix:

$$f'(q) = \begin{bmatrix} \partial\mu/\partial h & \partial\mu/\partial\mu \\ \partial\phi/\partial h & \partial\phi/\partial\mu \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix},$$

Eigenvalues:

$$\lambda^1 = u - \sqrt{gh}, \qquad \lambda^2 = u + \sqrt{gh}.$$

Eigenvectors:

$$r^1 = \begin{bmatrix} 1 \\ u - \sqrt{gh} \end{bmatrix}, \qquad r^2 = \begin{bmatrix} 1 \\ u + \sqrt{gh} \end{bmatrix}.$$

Notes:

Shallow water equations

Hydrostatic pressure:

Pressure at depth $z > 0$ below the surface is gz from weight of water above.

Depth-averaged pressure is

$$p = \int_0^h gz\,dz = \frac{1}{2}gz^2\Big|_0^h = \frac{1}{2}gh^2.$$

Notes:

Compressible gas dynamics

In one space dimension (e.g. in a pipe).
$\rho(x,t)$ = density, $u(x,t)$ = velocity,
$p(x,t)$ = pressure, $\rho(x,t)u(x,t)$ = momentum.

Conservation of:

mass:	ρ	flux:	ρu
momentum:	ρu	flux:	$(\rho u)u + p$
(energy)			

Conservation laws:

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$

Equation of state:

$$p = P(\rho).$$

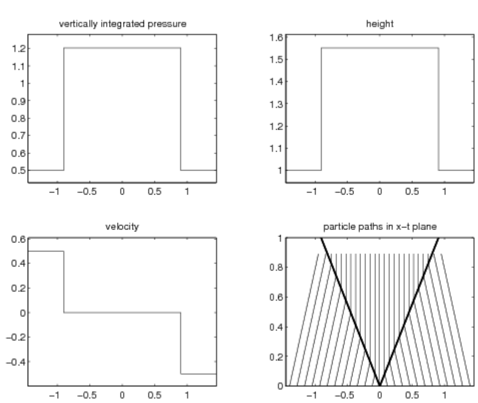
(Later: p may also depend on internal energy / temperature)

Notes:

Two-shock Riemann solution for shallow water

Initially $h_l = h_r = 1, \quad u_l = -u_r = 0.5 > 0$

Solution at later time:

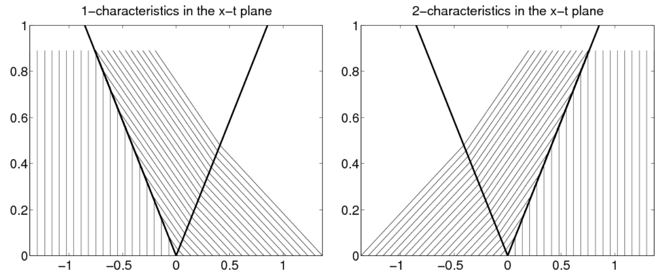


Notes:

Two-shock Riemann solution for shallow water

Characteristic curves $X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)}$

Slope of characteristic is constant in regions where q is constant. (Shown for $g = 1$ so $\sqrt{gh} = 1$ everywhere initially.)



Note that 1-characteristics impinge on 1-shock,
2-characteristics impinge on 2-shock.

Notes:

R.J. LeVeque, University of Washington IPDE 2011, July 6, 2011 [FVMHP Fig. 13.8]

An isolated shock

If an isolated shock with left and right states q_l and q_r is propagating at speed s

then the Rankine-Hugoniot condition must be satisfied:

$$f(q_r) - f(q_l) = s(q_r - q_l)$$

For a system $q \in \mathbb{R}^m$ this can only hold for certain pairs q_l, q_r:

For a linear system, $f(q_r) - f(q_l) = Aq_r - Aq_l = A(q_r - q_l)$.
So $q_r - q_l$ must be an eigenvector of $f'(q) = A$.

$A \in \mathbb{R}^{m\times m} \implies$ there will be m rays through q_l in state space in the eigen-directions, and q_r must lie on one of these.

For a nonlinear system, there will be m curves through q_l called the Hugoniot loci.

Notes:

R.J. LeVeque, University of Washington IPDE 2011, July 6, 2011 [FVMHP Fig. 13.7]

Hugoniot loci for shallow water

$$q = \begin{bmatrix} h \\ hu \end{bmatrix}, \qquad f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}.$$

Fix $q_* = (h_*, u_*)$.

What states q can be connected to q_* by an isolated shock?

The Rankine-Hugoniot condition $s(q - q_*) = f(q) - f(q_*)$ gives:

$$s(h_* - h) = h_*u_* - hu,$$
$$s(h_*u_* - hu) = h_*u_*^2 - hu^2 + \frac{1}{2}g(h_*^2 - h^2).$$

Two equations with 3 unknowns (h, u, s), so we expect 1-parameter families of solutions.

Notes:

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Hugoniot loci for shallow water

Rankine-Hugoniot conditions:

$$s(h_* - h) = h_*u_* - hu,$$
$$s(h_*u_* - hu) = h_*u_*^2 - hu^2 + \frac{1}{2}g(h_*^2 - h^2).$$

For any $h > 0$ we can solve for

$$u(h) = u_* \pm \sqrt{\frac{g}{2}\left(\frac{h_*}{h} - \frac{h}{h_*}\right)(h_* - h)}$$
$$s(h) = (h_*u_* - hu)/(h_* - h).$$

This gives 2 curves in h–hu space (one for $+$, one for $-$).

Notes:

Hugoniot loci for shallow water

For any $h > 0$ we have a possible shock state. Set

$$h = h_* + \alpha,$$

so that $h = h_*$ at $\alpha = 0$, to obtain

$$hu = h_*u_* + \alpha\left[u_* \pm \sqrt{gh_* + \frac{1}{2}g\alpha(3 + \alpha/h_*)}\right].$$

Hence we have

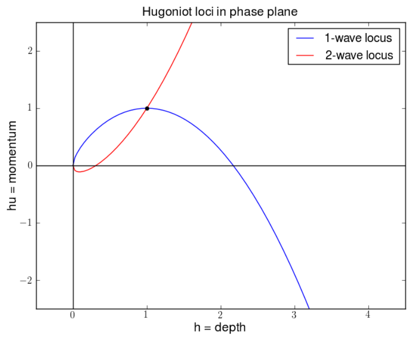
$$\begin{bmatrix} h \\ hu \end{bmatrix} = \begin{bmatrix} h_* \\ h_*u_* \end{bmatrix} + \alpha\begin{bmatrix} 1 \\ u_* \pm \sqrt{gh_* + \mathcal{O}(\alpha)} \end{bmatrix} \quad \text{as } \alpha \to 0.$$

Close to q_* the curves are tangent to eigenvectors of $f'(q_*)$
Expected since $f(q) - f(q_*) \approx f'(q_*)(q - q_*)$.

Notes:

Hugoniot loci for one particular q_*

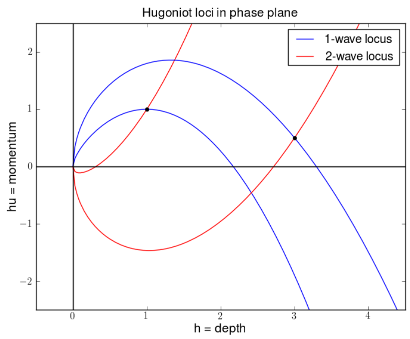
States that can be connected to q_* by a "shock"



Note: Might not satisfy entropy condition.

Notes:

Hugoniot loci for two different states

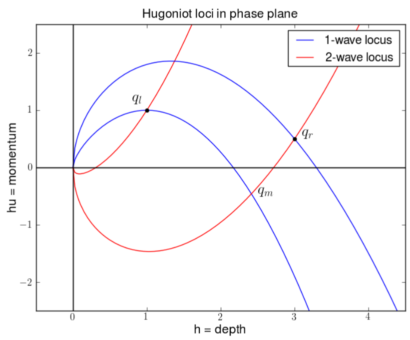


“All-shock” Riemann solution:

From q_l along 1-wave locus to q_m,
From q_r along 2-wave locus to q_m,

Notes:

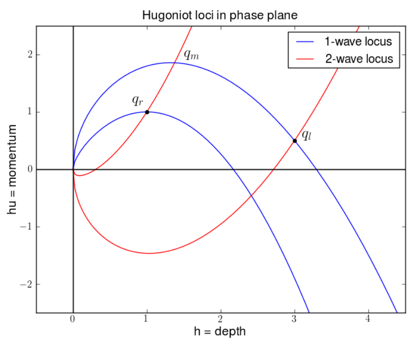
All-shock Riemann solution



From q_l along 1-wave locus to q_m,
From q_r along 2-wave locus to q_m,

Notes:

All-shock Riemann solution



From q_l along 1-wave locus to q_m,
From q_r along 2-wave locus to q_m,

Notes:

2-shock Riemann solution for shallow water

Given arbitrary states q_l and q_r, we can solve the Riemann problem with two shocks.

Choose q_m so that q_m is on the 1-Hugoniot locus of q_l and also q_m is on the 2-Hugoniot locus of q_r.

This requires

$$u_m = u_r + (h_m - h_r)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_r}\right)}$$

and

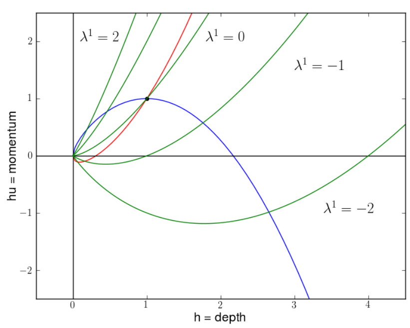
$$u_m = u_l - (h_m - h_l)\sqrt{\frac{g}{2}\left(\frac{1}{h_m} + \frac{1}{h_l}\right)}.$$

Equate and solve single nonlinear equation for h_m.

Notes:

Hugoniot loci for one particular q_*

Green curves are contours of λ^1

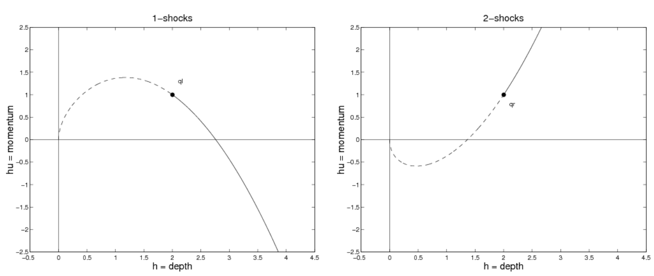


Note: Increases in one direction only along blue curve.

Notes:

Hugoniot locus for shallow water

States that can be connected to the given state by a 1-wave or 2-wave satisfying the R-H conditions:



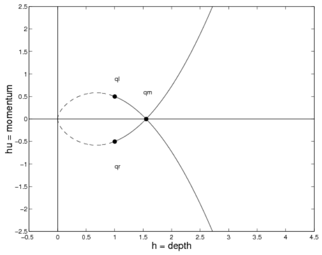
Solid portion: states that can be connected by shock satisfying entropy condition.

Dashed portion: states that can be connected with R-H condition satisfied but not the physically correct solution.

Notes:

2-shock Riemann solution for shallow water

Colliding with $u_l = -u_r > 0$:



Entropy condition: Characteristics should impinge on shock:

- λ^1 should decrease going from q_l to q_m,
- λ^2 should increase going from q_r to q_m,

This is satisfied along solid portions of Hugoniot loci above, not satisfied on dashed portions (entropy-violating shocks).

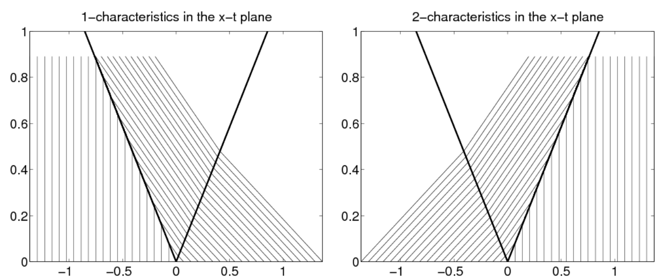
Notes:

R.J. LeVeque, University of Washington IPDE 2011, July 6, 2011 [FVMHP Fig. 13.10]

Two-shock Riemann solution for shallow water

Characteristic curves $X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)}$

Slope of characteristic is constant in regions where q is constant. (Shown for $g = 1$ so $\sqrt{gh} = 1$ everywhere initially.)



Note that 1-characteristics impinge on 1-shock, 2-characteristics impinge on 2-shock.

Notes:

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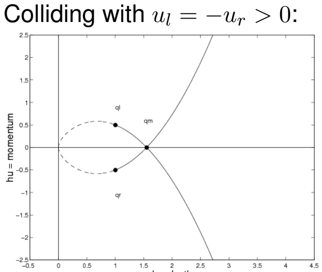
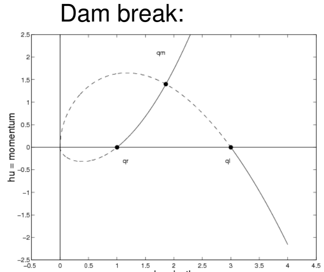
2-shock Riemann solution for shallow water

Colliding with $u_l = -u_r > 0$:



Dam break:



Entropy condition: Characteristics should impinge on shock:

- λ^1 should decrease going from q_l to q_m,
- λ^2 should increase going from q_r to q_m,

This is satisfied along solid portions of Hugoniot loci above, not satisfied on dashed portions (entropy-violating shocks).

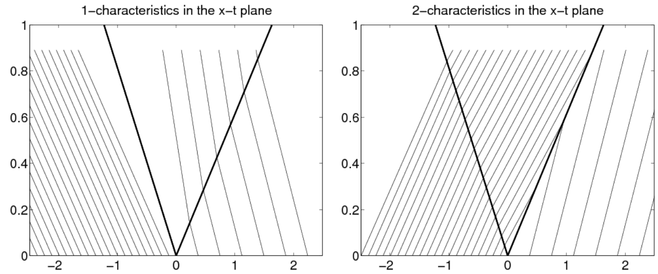
Notes:

R.J. LeVeque, University of Washington IPDE 2011, July 6, 2011 [FVMHP Fig. 13.10]

Entropy-violatiing Riemann solution for dam break

Characteristic curves $X'(t) = u(X(t), t) \pm \sqrt{gh(X(t), t)}$

Slope of characteristic is constant in regions where q is constant.



Note that 1-characteristics do not impinge on 1-shock, 2-characteristics impinge on 2-shock.

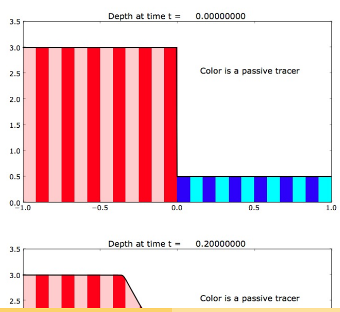
Notes:

R.J. LeVeque, University of Washington IPDE 2011, July 6, 2011 [FVMHP Fig. 13.11]

The Riemann problem

Dam break problem for shallow water equations

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



Notes:

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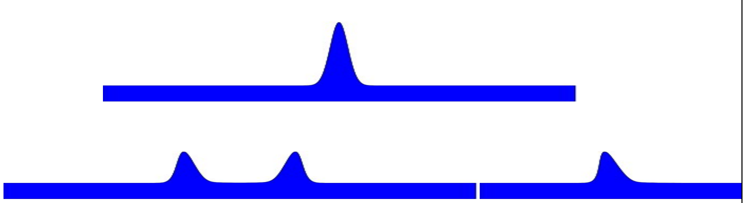
Simple waves



After separation, before shock formation:

Left- and right-going waves look like solutions to scalar equation.

Simple waves: q varies along an integral curve of $r^p(q)$.

Notes:

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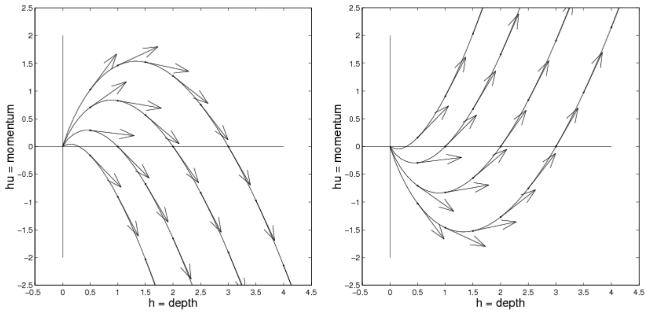
Integral curves of r^p

Curves in phase plane that are tangent to $r^p(q)$ at each q.



$\tilde{q}(\xi)$: curve through phase space parameterized by $\xi \in \mathbb{R}$.

Satisfying $\tilde{q}'(\xi) = \alpha(\xi) r^p(\tilde{q}(\xi))$ for some scalar $\alpha(\xi)$.

Notes:

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1-waves: integral curves of r^1

$\tilde{q}(\xi)$: curve through phase space parameterized by $\xi \in \mathbb{R}$.

Satisfies $\tilde{q}'(\xi) = \alpha(\xi) r^1(\tilde{q}(\xi))$ for some scalar $\alpha(\xi)$.

Choose $\alpha(\xi) \equiv 1$ and obtain

$$\begin{bmatrix} (\tilde{q}^1)' \\ (\tilde{q}^2)' \end{bmatrix} = \tilde{q}'(\xi) = r^1(\tilde{q}(\xi)) = \begin{bmatrix} 1 \\ \tilde{q}^2/\tilde{q}^1 - \sqrt{g\tilde{q}^1} \end{bmatrix}$$

This is a system of 2 ODEs

First equation: $\tilde{q}^1(\xi) = \xi \implies \xi = h$.
Second equation $\implies (\tilde{q}^2)' = \tilde{q}^2(\xi)/\xi - \sqrt{g\xi}$.

Require $\tilde{q}^2(h_*) = h_* u_* \implies$

$$\tilde{q}^2(\xi) = \xi u_* + 2\xi\left(\sqrt{gh_*} - \sqrt{g\xi}\right).$$

Notes:

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1-wave integral curves of r^p

So

$$\tilde{q}^1(\xi) = \xi,$$
$$\tilde{q}^2(\xi) = \xi u_* + 2\xi\left(\sqrt{gh_*} - \sqrt{g\xi}\right).$$

and hence

$$hu = hu_* + 2h\left(\sqrt{gh_*} - \sqrt{gh}\right).$$

Similarly, 2-wave integral curves satisfy

$$hu = hu_* - 2h\left(\sqrt{gh_*} - \sqrt{gh}\right).$$

Notes:

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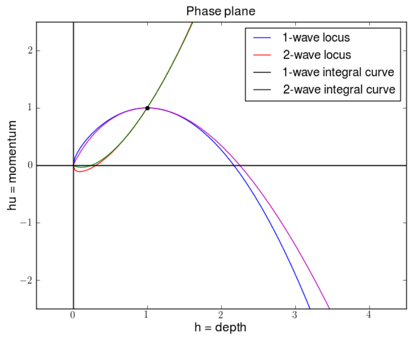
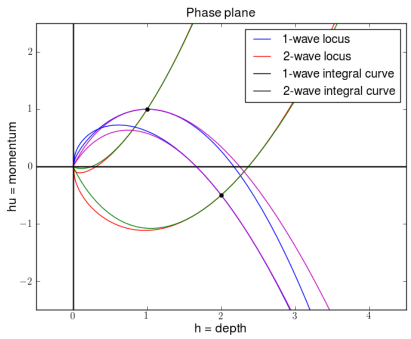
Integral curves of r^p versus Hugoniot loci



Notes:

Integral curves of r^p versus Hugoniot loci



Solution to Riemann problem depends on which state is q_l, q_r.

Also need to choose correct curve from each state.

Notes: