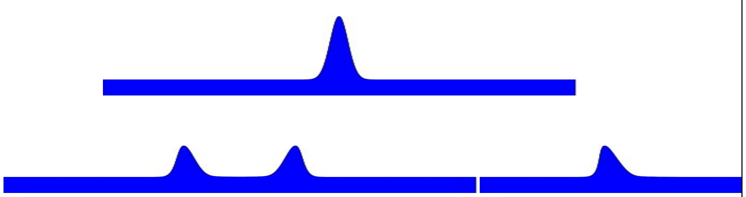
Outline

- Simple waves, rarefaction waves
- Integral curves in phase plane
- Approximate Riemann solvers
- Dam break and tsunami modeling
- Adaptive mesh refinement

Notes:

Simple waves



After separation, before shock formation:

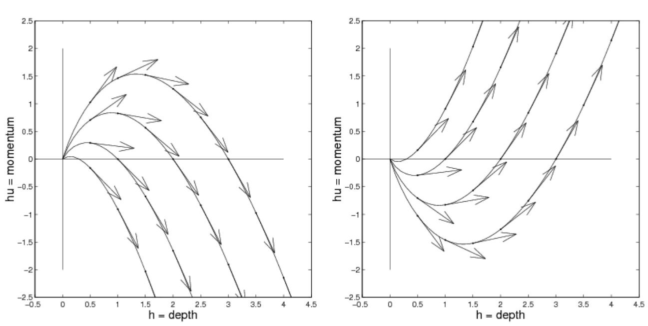
Left- and right-going waves look like solutions to scalar equation.

Simple waves: q varies along an integral curve of $r^p(q)$.

Notes:

Integral curves of r^p

Curves in phase plane that are tangent to $r^p(q)$ at each q.

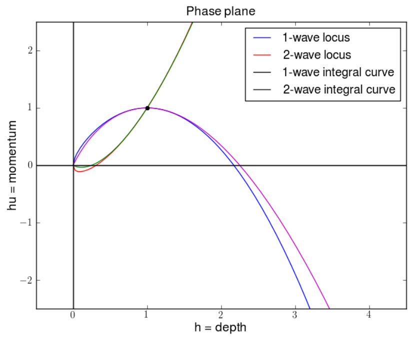


$\tilde{q}(\xi)$: curve through phase space parameterized by $\xi \in \mathbb{R}$.

Satisfying $\tilde{q}'(\xi) = \alpha(\xi) r^p(\tilde{q}(\xi))$ for some scalar $\alpha(\xi)$.

Notes:

Integral curves of r^p versus Hugoniot loci



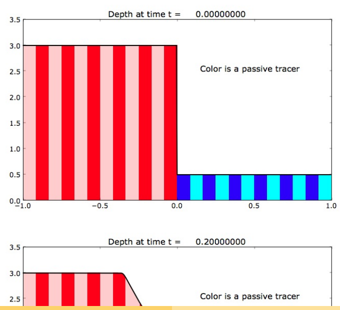
Notes:

The Riemann problem

Dam break problem for shallow water equations

$$h_t + (hu)_x = 0$$

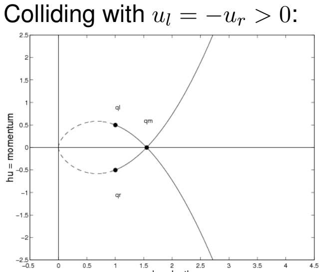
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



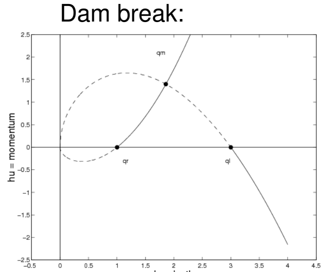
Notes:

2-shock Riemann solution for shallow water

Colliding with $u_l = -u_r > 0$:



Dam break:



Entropy condition: Characteristics should impinge on shock:
λ^1 should decrease going from q_l to q_m,
λ^2 should increase going from q_r to q_m,

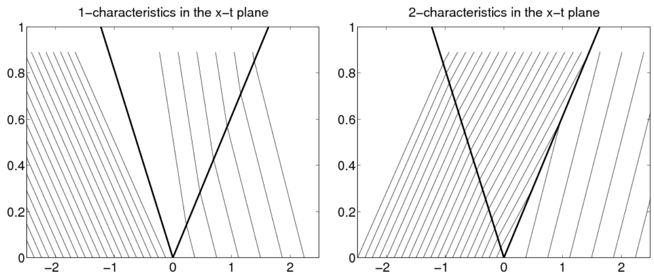
This is satisfied along solid portions of Hugoniot loci above, not satisfied on dashed portions (entropy-violating shocks).

Notes:

Entropy-violatiing Riemann solution for dam break

Characteristic curves $X'(t) = u(X(t),t) \pm \sqrt{gh(X(t),t)}$

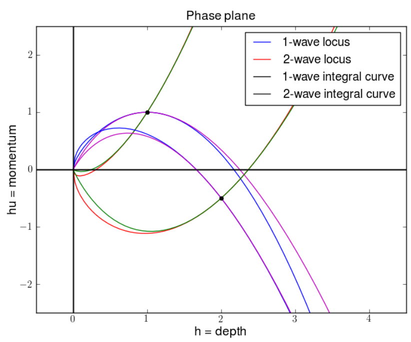
Slope of characteristic is constant in regions where q is constant.



Note that 1-characteristics do not impinge on 1-shock, 2-characteristics impinge on 2-shock.

Notes:

Integral curves of r^p versus Hugoniot loci



Solution to Riemann problem depends on which state is q_l, q_r.

Also need to choose correct curve from each state.

Notes:

Rarefaction waves

Centered rarefaction wave:

Similarity solution with piecewise constant initial data:

$$q(x,t) = \begin{cases} q_l & \text{if } x/t \le \xi_1 \\ \tilde{q}(x/t) & \text{if } \xi_1 \le x/t \le \xi_2 \\ q_r & \text{if } x/t \ge \xi_2, \end{cases}$$

where q_l and q_r are two points on a single integral curve with $\lambda^p(q_l) < \lambda^p(q_r)$.

Required so that characteristics spread out as time advances.

Also want $\lambda^p(q)$ monotonically increasing from q_l to q_r.

This genuine nonlinearity generalizes convexity of scalar flux.

Notes:

Genuine nonlinearity

For scalar problem $q_t + f(q)_x = 0$, want $f''(q) \neq 0$ everywhere.

This implies that $f'(q)$ is monotonically increasing or decreasing between q_l and q_r.

Shock if decreasing, Rarefaction if increasing.

For system we want $\lambda^p(q)$ to be monotonically varying along integral curve of $r^p(q)$.

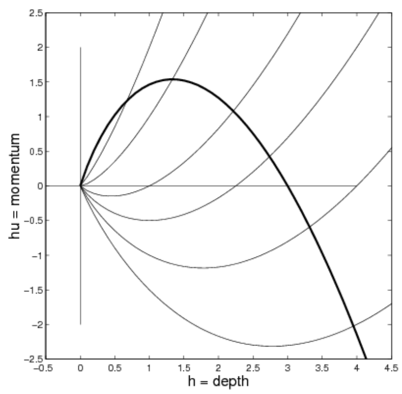
If so then this field is genuinely nonlinear.

This requires $\nabla\lambda^p(q) \cdot r^p(q) \neq 0$.

Notes:

Genuine nonlinearity of shallow water equations

Integral curves (heavy line) and contours of λ^1:



Notes:

Genuine nonlinearity of shallow water equations

1-waves: Requires $\nabla\lambda^1(q) \cdot r^1(q) \neq 0$.

$$\lambda^1 = u - \sqrt{gh} = q^2/q^1 - \sqrt{gq^1},$$

$$\nabla\lambda^1 = \begin{bmatrix} -q^2/(q^1)^2 - \frac{1}{2}\sqrt{g/q^1} \\ 1/q^1 \end{bmatrix},$$

$$r^1 = \begin{bmatrix} 1 \\ q^2/q^1 - \sqrt{gq^1} \end{bmatrix},$$

and hence

$$\nabla\lambda^1 \cdot r^1 = -\frac{3}{2}\sqrt{g/q^1} = -\frac{3}{2}\sqrt{g/h}$$
$$< 0 \quad \text{for all} \quad h > 0.$$

Notes:

Linearly degenerate fields

Scalar advection: $q_t + uq_x = 0$ with $u =$ constant.

Characteristics $X(t) = x_0 + ut$ are parallel.

Discontinuity propagates along a characteristic curve.

Characteristics on either side are parallel so not a shock!

For system the analogous property arises if

$$\nabla\lambda^p(q) \cdot r^p(q) \equiv 0$$

holds for all q, in which case
λ^p is constant along each integral curve.

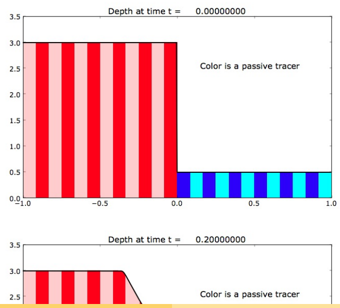
Then pth field is said to be linearly degenerate.

Notes:

The Riemann problem

Dam break problem for shallow water equations

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0$$



Notes:

Shallow water with passive tracer

Let $\phi(x,t)$ be tracer concentration and add equation

$$\phi_t + u\phi_x = 0 \implies (h\phi)_t + (uh\phi)_x = 0.$$

Gives:

$$q = \begin{bmatrix} h \\ hu \\ h\phi \end{bmatrix} = \begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix}, \quad f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ uh\phi \end{bmatrix} = \begin{bmatrix} q^2 \\ (q^2)/q^1 + \frac{1}{2}g(q^1)^2 \\ q^2q^3/q^1 \end{bmatrix}.$$

Jacobian:

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -u\phi & \phi & u \end{bmatrix}.$$

Notes:

Shallow water with passive tracer

$$f'(q) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & 0 \\ -u\phi & \phi & u \end{bmatrix}.$$

$$\lambda^1 = u - \sqrt{gh}, \qquad \lambda^2 = u, \qquad \lambda^3 = u + \sqrt{gh},$$

$$r^1 = \begin{bmatrix} 1 \\ u - \sqrt{gh} \\ \phi \end{bmatrix}, \quad r^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad r^3 = \begin{bmatrix} 1 \\ u + \sqrt{gh} \\ \phi \end{bmatrix}.$$

$$\lambda^2 = u = (hu)/h \implies \nabla\lambda^2 = \begin{bmatrix} -u/h \\ 1/h \\ 0 \end{bmatrix} \implies \lambda^2 \cdot r^2 \equiv 0.$$

So 2nd field is linearly degenerate.
(Fields 1 and 3 are genuinely nonlinear.)

Notes:

Euler equations of gas dynamics

Conservation of mass, momentum, energy: $q_t + f(q)_x = 0$ with

$$q = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \qquad f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}$$

where

$$p = \text{pressure} = p(\rho, E) \quad \text{(Equation of state)}$$

The Jacobian $f'(q)$ has eigenvalues $u - c, \;\; u, \;\; u + c$ where

$$c = \sqrt{\frac{dp}{d\rho}} \text{ at constant entropy}$$

Eigenvalues vary with $q \implies$ shocks, rarefactions.

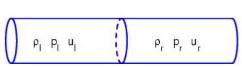
Notes:

Riemann Problem for Euler equations

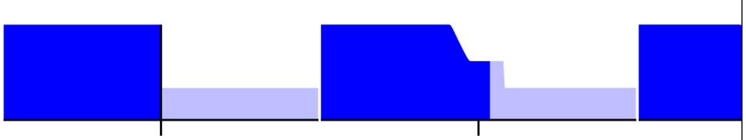
Initial data:

$$q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

Shock tube problem: $u_l = u_r = 0$, jump in ρ and p.



Pressure:

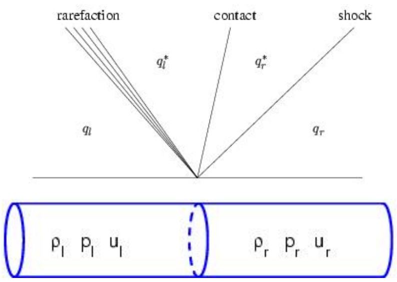


This is also solution to dam break problem for shallow water equations.

Notes:

Riemann Problem for gas dynamics

Waves propagating in x–t space:



Similarity solution (function of x/t alone).

Waves can be approximated by discontinuties:

Approximate Riemann solvers

Notes:

Approximate Riemann Solvers

Approximate true Riemann solution by set of waves consisting of finite jumps propagating at constant speeds.

Local linearization:

Replace $q_t + f(q)_x = 0$ by

$$q_t + \hat{A} q_x = 0,$$

where $\hat{A} = \hat{A}(q_l, q_r) \approx f'(q_{ave})$.

Then decompose

$$q_r - q_l = \alpha^1 \hat{r}^1 + \cdots \alpha^m \hat{r}^m$$

to obtain waves $\mathcal{W}^p = \alpha^p \hat{r}^p$ with speeds $s^p = \hat{\lambda}^p$.

Notes:

Approximate Riemann Solvers

How to use?

One approach: determine $Q^* =$ state along $x/t = 0$,

$$Q^* = Q_{i-1} + \sum_{p:s^p<0} \mathcal{W}^p, \quad F_{i-1/2} = f(Q^*),$$

$$\mathcal{A}^- \Delta Q = F_{i-1/2} - f(Q_{i-1}), \quad \mathcal{A}^+ \Delta Q = f(Q_i) - F_{i-1/2}.$$

Or, sometimes can use:

$$\mathcal{A}^- \Delta Q = \sum_{p:s^p<0} s^p \mathcal{W}^p, \qquad \mathcal{A}^+ \Delta Q = \sum_{p:s^p>0} s^p \mathcal{W}^p.$$

Conservative only if $\mathcal{A}^- \Delta Q + \mathcal{A}^+ \Delta Q = f(Q_i) - f(Q_{i-1})$.

This holds for Roe solver.

Notes:

Roe Solver

Solve $q_t + \hat{A} q_x = 0$ where $\hat{A}$ satisfies

$$\hat{A}(q_r - q_l) = f(q_r) - f(q_l).$$

Then:

- Good approximation for weak waves (smooth flow)
- Single shock captured exactly:

  $f(q_r) - f(q_l) = s(q_r - q_l) \implies q_r - q_l$ is an eigenvector of $\hat{A}$
- Wave-propagation algorithm is conservative since

  $$\mathcal{A}^- \Delta Q_{i-1/2} + \mathcal{A}^+ \Delta Q_{i-1/2} = \sum s^p_{i-1/2} \mathcal{W}^p_{i-1/2} = A \sum \mathcal{W}^p_{i-1/2}.$$

Roe average $\hat{A}$ can be determined analytically for many important nonlinear systems (e.g. Euler, shallow water)

Notes:

Shallow water equations

$h(x,t) =$ depth
$u(x,t) =$ velocity (depth averaged, varies only with x)

Conservation of mass and momentum hu gives system of two equations.

mass flux $= hu$,
momentum flux $= (hu)u + p$ where $p =$ hydrostatic pressure

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2} g h^2\right)_x = 0$$

Jacobian matrix:

$$f'(q) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix}, \qquad \lambda = u \pm \sqrt{gh}.$$

Notes:

Roe solver for Shallow Water

Given $h_l,\ u_l,\ h_r,\ u_r$, define

$$\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l} u_l + \sqrt{h_r} u_r}{\sqrt{h_l} + \sqrt{hr}}$$

Then

$\hat{A} =$ Jacobian matrix evaluated at this average state

satisfies

$$A(q_r - q_l) = f(q_r) - f(q_l).$$

- Roe condition is satisfied,
- Isolated shock modeled well,
- Wave propagation algorithm is conservative,
- High resolution methods obtained using corrections with limited waves.

Notes:

Roe solver for Shallow Water

Given $h_l,\ u_l,\ h_r,\ u_r$, define

$$\bar{h} = \frac{h_l + h_r}{2}, \quad \hat{u} = \frac{\sqrt{h_l}u_l + \sqrt{h_r}u_r}{\sqrt{h_l} + \sqrt{hr}}$$

Eigenvalues of $\hat{A} = f'(\hat{q})$ are:

$$\hat{\lambda}^1 = \hat{u} - \hat{c}, \quad \hat{\lambda}^2 = \hat{u} + \hat{c}, \quad \hat{c} = \sqrt{g\bar{h}}.$$

Eigenvectors:

$$\hat{r}^1 = \begin{bmatrix} 1 \\ \hat{u} - \hat{c} \end{bmatrix}, \qquad \hat{r}^2 = \begin{bmatrix} 1 \\ \hat{u} + \hat{c} \end{bmatrix}.$$

See $CLAW/examples/shallow/1d.

Notes:

HLL Solver

Harten – Lax – van Leer (1983): Use only 2 waves with
s^1 =minimum characteristic speed
s^2 =maximum characteristic speed

$$\mathcal{W}^1 = Q^* - q_l, \qquad \mathcal{W}^2 = q_r - Q^*$$

Conservation implies unique value for middle state Q^*:

$$s^1\mathcal{W}^1 + s^2\mathcal{W}^2 = f(q_r) - f(q_l)$$

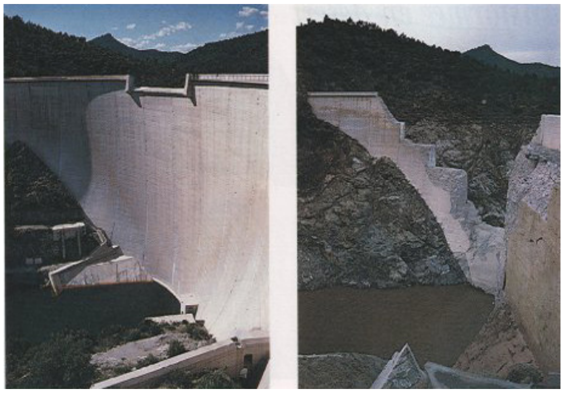
$$\implies Q^* = \frac{f(q_r) - f(q_l) - s^2 q_r + s^1 q_l}{s^1 - s^2}.$$

Einfeldt (HLLE): Formulas for speeds in gas dynamics based on characteristic speeds and Roe averages that gives positivity.

Notes:

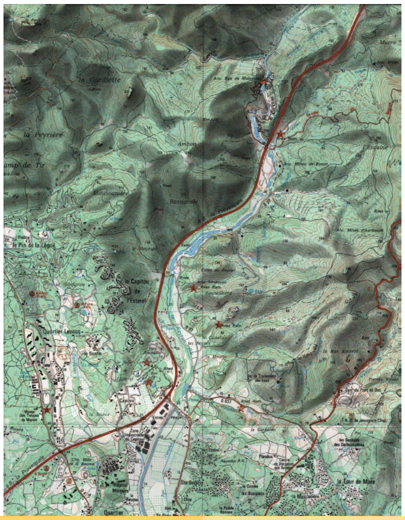
Malpasset Dam Failure

Catastrophic failure in 1959



Notes:

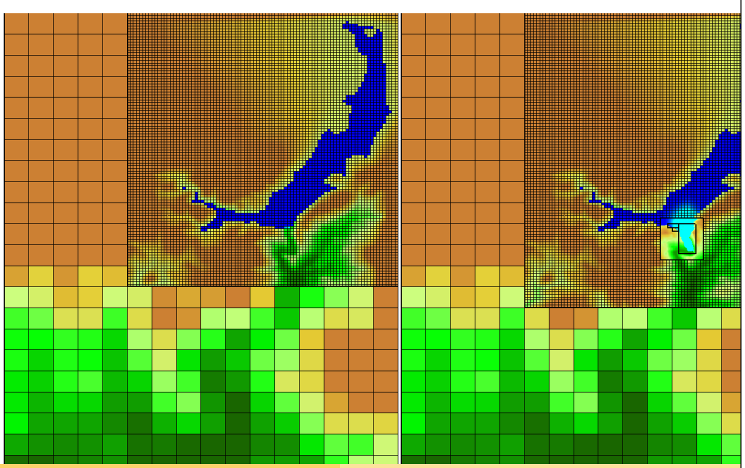
Malpasset Dam Failure



Notes:

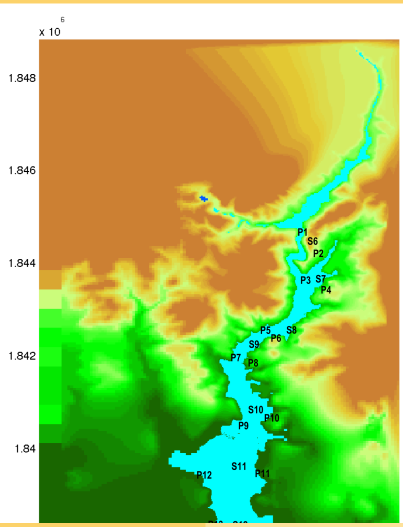
Modeling work by David George, using GeoClaw

Coarse: 400m cell side, Level 2: 50m, Level 3: 12m, Level 4: 3m



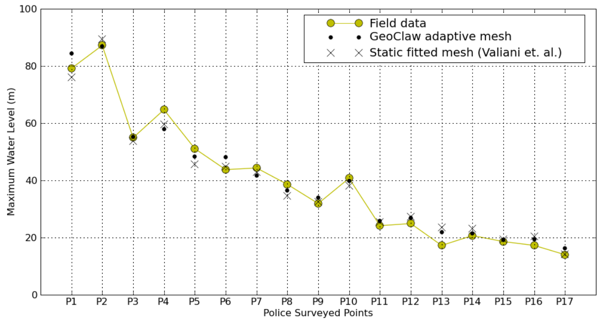
Notes:

Malpasset survey locations



Notes:

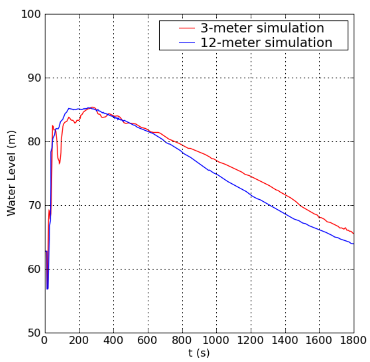
Malpasset survey locations



Notes:

Grid convergence study

Water depth gauge at location P2 computed with two different resolutions (using 4 levels or only 3):



Notes:

Tsunamis

Generated by

- Earthquakes,
- Landslides,
- Submarine landslides,
- Volcanoes,
- Meteorite or asteroid impact

There were 97 significant tsunamis during the 1990's, causing 16,000 casualties.

There have been approximately 28 tsunamis with run-up greater than 1m on the west coast of the U.S. since 1812.

Notes:

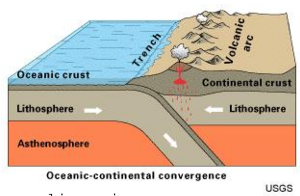
Sumatra event of December 26, 2004

Magnitude 9.1 quake near Sumatra, where Indian tectonic plate is being subducted under the Burma platelet.

Rupture along subduction zone
$\approx$ 1200 km long, 150 km wide

Propagating at $\approx$ 2 km/sec (for $\approx$ 10 minutes)

Fault slip up to 15 m, uplift of several meters.
(Fault model from Caltech Seismolab.)



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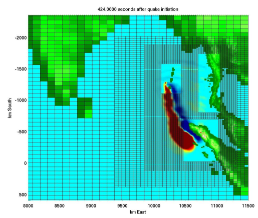


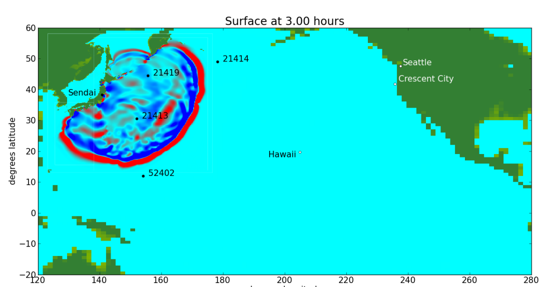
Notes:

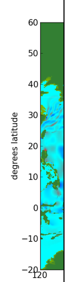
Tsunamis

- Small amplitude in ocean (< 1 meter) but can grow to 10s of meters at shore.
- Run-up along shore can inundate 100s of meters inland
- Long wavelength (as much as 200 km)
- Propagation speed $\sqrt{gh}$ (much slower near shore)
- Average depth of Pacific or Indian Ocean is 4000 m
  $\Longrightarrow$ average speed 200 m/s $\approx$ 450 mph

Notes:

Great Tohoku Tsunami, 11 March 2011





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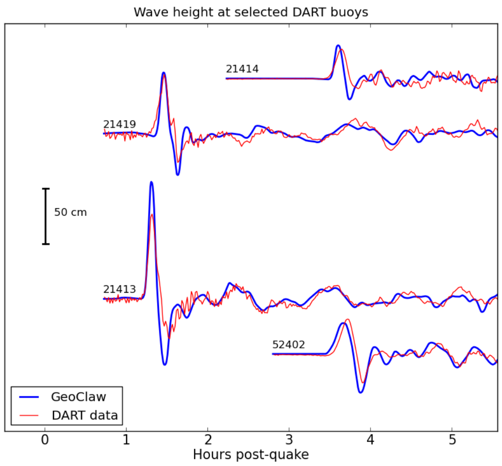
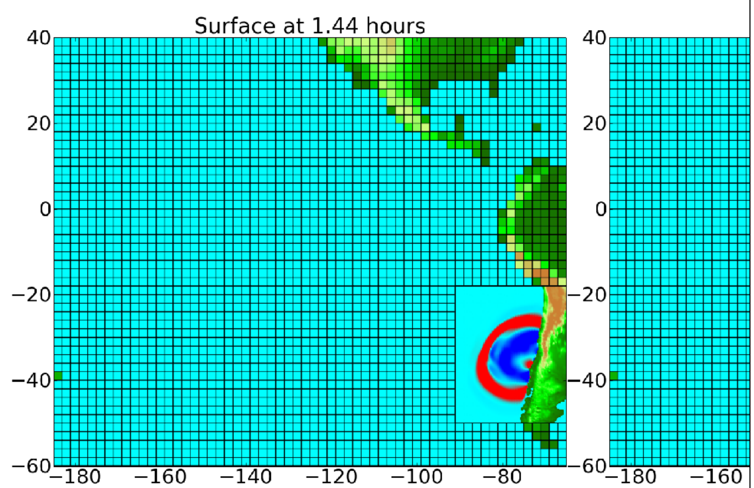


Notes:

27 February 2010 tsunami



Notes:

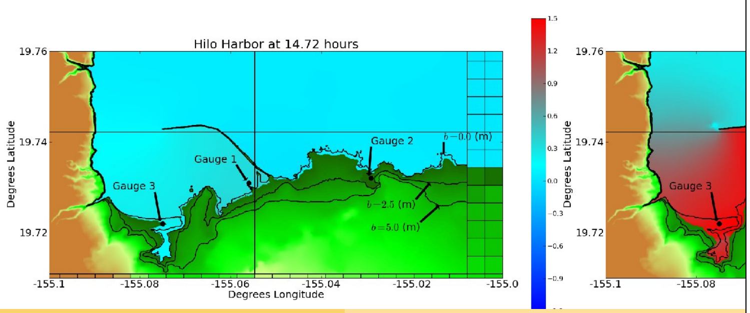
Inundation of Hilo, Hawaii

Using 5 levels of refinement with ratios 8, 4, 16, 32.

Resolution ≈ 160 km on Level 1 and ≈ 10m on Level 5.

Total refinement factor: $2^{14} = 16,384$ in each direction.

With 15 m displacement at fault:
With 90 m displacement at fault:



Notes:

Shallow water equations with bathymetry $B(x, y)$

$$
\begin{aligned}
h_t + (hu)_x + (hv)_y &= 0 \\
(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= -ghB_x(x, y) \\
(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= -ghB_y(x, y)
\end{aligned}
$$

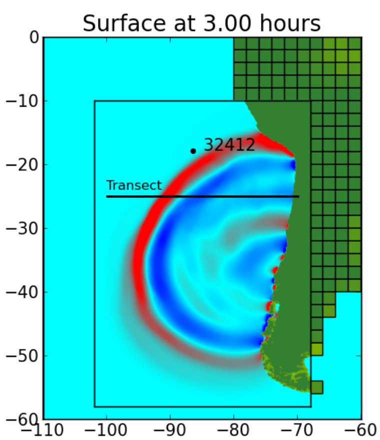
Some issues:

- Delicate balance between flux divergence and bathymetry:
  h varies on order of 4000m, rapid variations in ocean
  Waves have magnitude 1m or less.
- Cartesian grid used, with $h = 0$ in dry cells:
  Cells become wet/dry as wave advances on shore
  Robust Riemann solvers needed.
- Adaptive mesh refinement crucial
  Interaction of AMR with source terms, dry states

Notes:

R.J. LeVeque, University of Washington IPDE 2011, July 7, 2011

Tsunami from 27 Feb 2010 quake off Chile



R.J. LeVeque, University of Washington IPDE 2011, July 7, 2011

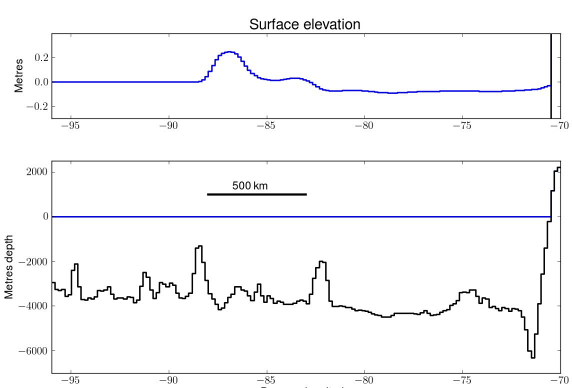
Notes:

R.J. LeVeque, University of Washington IPDE 2011, July 7, 2011

Transect of 27 February 2010 tsunami

Bathymetry, depth change by > 1000 m from one cell to next,

Surface elevation changes on order of a few cm.



R.J. LeVeque, University of Washington IPDE 2011, July 7, 2011

Notes:

R.J. LeVeque, University of Washington IPDE 2011, July 7, 2011

Source terms and quasi-steady solutions

$$q_t + f(q)_x = \psi(q)$$

Steady-state solution:

$$q_t = 0 \implies f(q)_x = \psi(q)$$

Balance between flux gradient and source.

Quasi-Steady solution:

Small perturbation propagating against steady-state background.

$$q_t \ll f(q)_x \approx \psi(q)$$

Want accurate calculation of perturbation.

Examples:

- Shallow water equations with bottom topography and flat surface
- Stationary atmosphere where pressure gradient balances gravity

Notes:

Fractional steps for a quasisteady problem

Alternate between solving homogeneous conservation law

$$q_t + f(q)_x = 0 \tag{1}$$

and source term

$$q_t = \psi(q). \tag{2}$$

When $q_t \ll f(q)_x \approx \psi(q)$:

- Solving (**??**) gives large change in q
- Solving (**??**) should essentially cancel this change.

Numerical difficulties:

- (**??**) and (**??**) are solved by very different methods. Generally will not have proper cancellation.
- Nonlinear limiters are applied to $f(q)_x$ term, not to small-perturbation waves. Large variation in steady state hides structure of waves.

Notes:

Incorporating source term in f-waves

$q_t + f(q)_x = \psi$ with $f(q)_x \approx \psi$.

Concentrate source at interfaces: $\Psi_{i-1/2}\,\delta(x - x_{i-1/2})$

Split $f(Q_i) - f(Q_{i-1}) - \Delta x \Psi_{i-1/2} = \sum_p \mathcal{Z}^p_{i-1/2}$

Use these waves in wave-propagation algorithm.

Steady state maintained: (Well balanced)

If $\frac{f(Q_i)-f(Q_{i-1})}{\Delta x} = \Psi_{i-1/2}$ then $\mathcal{Z}^p \equiv 0$

Near steady state:

Deviation from steady state is split into waves and limited.

Notes:

Incorporating source term in f-waves

$$q_t + f(q)_x = \psi(q)\sigma_x(x) \implies \Psi_{i-1/2}(\sigma_i - \sigma_{i-1})$$

Question: How to average $\psi(q)$ between cells to get $\Psi_{i-1/2}$?

A Well-Balanced Path-Integral f-wave Method for Hyperbolic Problems with Source Terms, to appear in *J. Sci. Comput.*

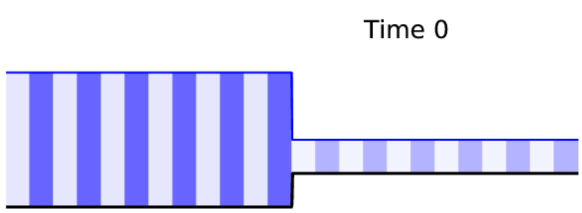
For some problems (e.g. ocean-at-rest) can simply use arithmetic average.

$$\Psi_{i-1/2} = \frac{1}{2}(\psi(Q_{i-1}) + \psi(Q_i)).$$

Notes:

The Riemann problem over topography

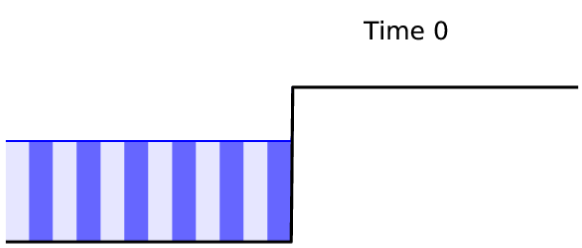
$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghB_x(x)$$



Notes:

The Riemann problem with dry state

$$h_t + (hu)_x = 0$$
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghB_x(x)$$

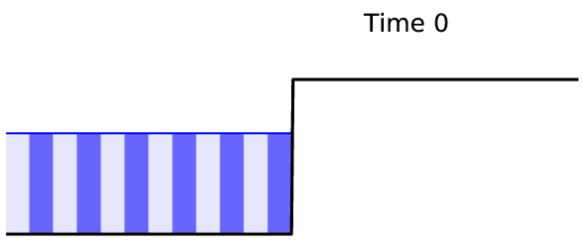


Notes:

The Riemann problem with dry state

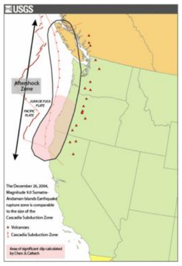
$$h_t + (hu)_x = 0$$

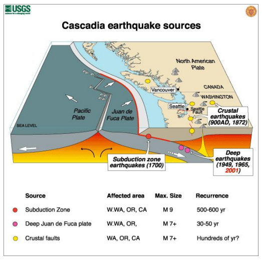
$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghB_x(x)$$



Notes:

Cascadia subduction fault





- 1200 km long off-shore fault stretching from northern California to southern Canada.
- Last major rupture: magnitude 9.0 earthquake on January 26, 1700.
- Tsunami recorded in Japan with run-up of up to 5 meters.
- Historically there appear to be magnitude 8 or larger quakes every 500 years on average.

Notes:

Tsunami Deposits

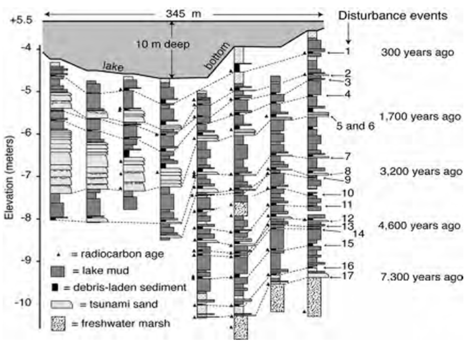


Figure 3.6. An example of long-term records of tsunami deposits interpreted to be from the Cascadia Subduction Zone; from Bradley Lake on the coast of southern Oregon (based on Kelsey et al., 2005).

From: J. Bourgeois, Chapter 3 of *The Sea, Volume 15: Tsunamis*, Harvard University Press, 2009.

Notes:

Cascadia event simulations

Magnitude 9.0 earthquake similar to 1700 event.

Dave Alexander, Bill Johnstone, SpatialVision, Vancouver, BC

Barbara Lence, Civil Engineering, UBC

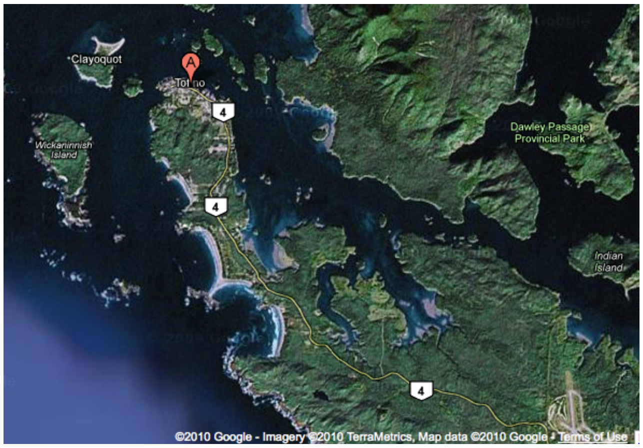
Movies:

Vancouver Island and Olympic Penninsula

Ucluelet

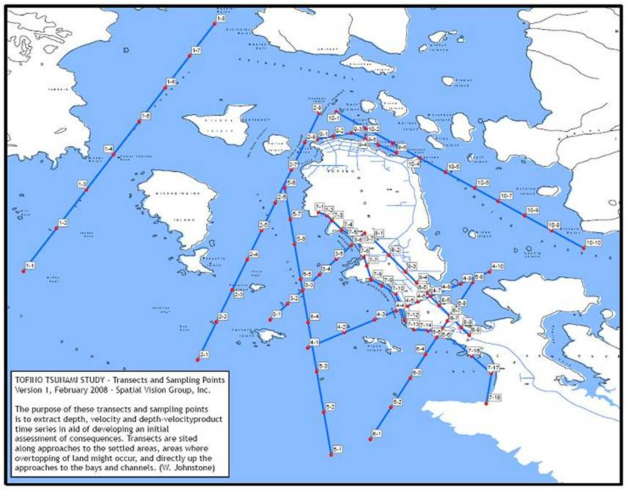
Notes:

Hazard Study for Tofino, BC



Notes:

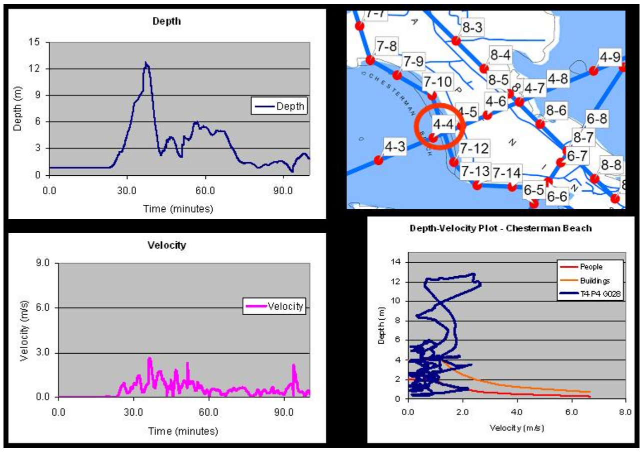
Hazard Study for Tofino, BC



q(1) at time 2299.2600

Notes:

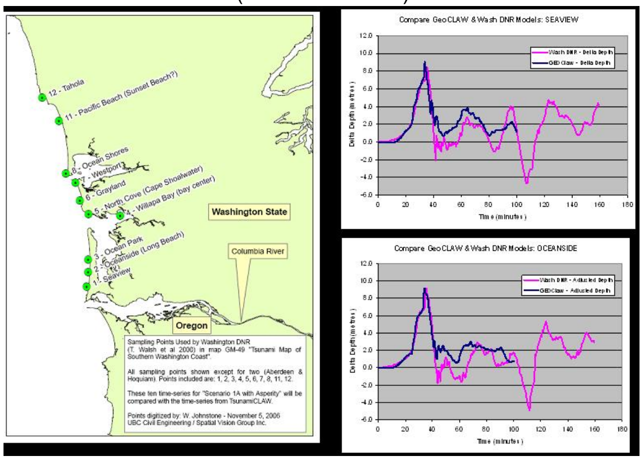
Hazard Study for Tofino, BC



Notes:

Comparison to NOAA model

Thanks to Tim Walsh (WA State DNR)



Notes: