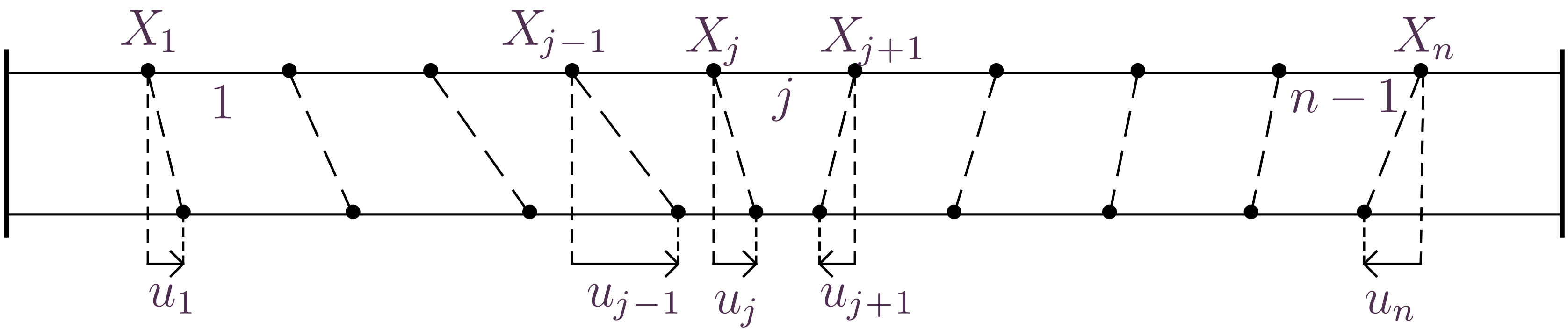
Lesson overview

- Continuum versus discrete mechanics
- Constitutive laws
- Tensor algebra, calculus
- Impact of machine learning

Consider an ensemble of linked springs



- Discrete point of view, for interior $j = 2, \ldots, n-1$, conservation of momentum

$$m_j \ddot{u}_j = k_j(u_{j+1} - u_j) - k_{j-1}(u_j - u_{j-1})$$

- Continuum point of view $\mathrm{d}m_j = \mu(x_j)\,\mathrm{d}x$, $\mathrm{d}k_j = \kappa(x_j)/\mathrm{d}x$, $u_j(t) = u(t, x_j)$

$$\mu\, u_{tt} = (\kappa\, u_x)_x \Rightarrow u_{tt} - c^2 u_{xx} = 0 \text{ for constant } \mu, \kappa, c^2 = \kappa / \mu$$

The basic ingredients of continuum mechanics

- Conservation laws (mass, momentum, energy)
- Definition of displacements $u(t,x)$
- Definition of forces $f(t,x)$
- Definition of a constitutive law linking forces to displacements, e.g., Hooke

$$f = \kappa u$$

Challenges in continuum mechanics

- 3D forces $\boldsymbol{f}(t, \boldsymbol{x})$ and displacements $\boldsymbol{u}(t, \boldsymbol{x})$
- Description in terms of:
  - original positions (Lagrangean) $\boldsymbol{u}(t, \boldsymbol{X})$
  - current position (Eulerian) $\boldsymbol{u}(t, \boldsymbol{x})$
  - current position in terms of original position $\boldsymbol{x}(t, \boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{u}(t, \boldsymbol{X})$
- Complicated calculus to evaluate deformation

$$\mathrm{d}\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x} + \mathrm{d}\boldsymbol{x}) - \boldsymbol{u}(t, \boldsymbol{x}) \cong \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \mathrm{d}\boldsymbol{x} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} \left(\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} \right)^{-1} \mathrm{d}\boldsymbol{X}$$

- Complicated constitutive laws: elastic, plastic, viscoelastic, anisotropic

Impact of machine learning

- Traditional continuum mechanics: simple constitutive hypotheses
  - Hookean elasticity
  - Maxwell viscoelasticity
  - Voigt viscoelasticity
  - Newtonian fluid
- Many materials of current interest deviate from above hypotheses
  - Challenge: how to approximate $\boldsymbol{f}(\boldsymbol{u})$ (high-dimensional approximation)
  - Empirical observations: deep neural networks are good approximations

$$\boldsymbol{f}(\boldsymbol{u}) \cong \boldsymbol{g}_1(\boldsymbol{g}_2(\ldots \boldsymbol{g}_L(\boldsymbol{u})))$$