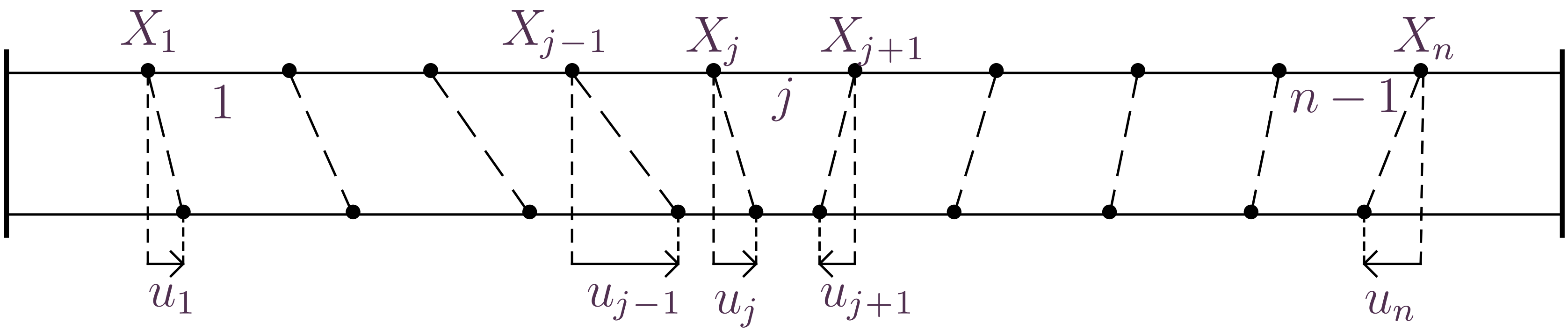
Lesson overview

- Continuum versus discrete mechanics
- Conservation laws
- Tensor algebra, calculus

Consider an ensemble of linked springs



- Discrete point of view, for interior $j = 2, \ldots, n-1$, conservation of momentum

$$m_j \ddot{u}_j = k_j(u_{j+1} - u_j) - k_{j-1}(u_j - u_{j-1})$$

- Continuum point of view $\mathrm{d}m_j = \mu(x_j)\,\mathrm{d}x$, $\mathrm{d}k_j = \kappa(x_j)/\mathrm{d}x$, $u_j(t) = u(t, x_j)$

$$\mu u_{tt} = (\kappa u_x)_x \Rightarrow u_{tt} - c^2 u_{xx} = 0 \text{ for constant } \mu, \kappa, c^2 = \kappa/\mu$$

- Continuum conservation laws have form $\partial_t \boldsymbol{q} = -\nabla \cdot \boldsymbol{f}(\boldsymbol{q})$, $\boldsymbol{f}$ is flux of $\boldsymbol{q}$
  - mass, e.g., continuity equation for fluid mechanics $\rho_t = -\nabla \cdot (\rho \boldsymbol{u})$
  - momentum, e.g., Newton's law $(\rho \boldsymbol{v})_t = -\nabla \cdot \boldsymbol{\sigma}$
  - energy, e.g., $E_t = -\nabla \cdot ((E + p)\, \boldsymbol{u})$

  Note that standard conventions for normal orientation differ:

  → Fluid mechanics uses inward pointing normals, hence $\partial_t \boldsymbol{q} = -\nabla \cdot \boldsymbol{f}(\boldsymbol{q})$

  → Solid mechanics uses outward pointing normals, hence $\partial_t \boldsymbol{q} = \nabla \cdot \boldsymbol{f}(\boldsymbol{q})$
- Definition of displacements $\boldsymbol{u}(t, \boldsymbol{x})$, or velocities $\boldsymbol{v}(t, \boldsymbol{x})$
- Definition of forces $\boldsymbol{f}(t, \boldsymbol{x})$
- Definition of a constitutive law linking forces to displacements, e.g., 1D Hooke

$$f = \kappa u$$

- 3D forces $\boldsymbol{f}(t,\boldsymbol{x})$ and displacements $\boldsymbol{u}(t,\boldsymbol{x})$
- Description in terms of:
  - original positions (Lagrangean) $\boldsymbol{u}(t,\boldsymbol{X})$
  - current position (Eulerian) $\boldsymbol{u}(t,\boldsymbol{x})$
  - current position in terms of original position $\boldsymbol{x}(t,\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{u}(t,\boldsymbol{X})$
- Complicated calculus to evaluate deformation

$$\mathrm{d}\boldsymbol{u} = \boldsymbol{u}(t,\boldsymbol{x}+\mathrm{d}\boldsymbol{x}) - \boldsymbol{u}(t,\boldsymbol{x}) \cong \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}\,\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\left(\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}\right)^{-1}\mathrm{d}\boldsymbol{X}$$

- Complicated constitutive laws: elastic, plastic, viscoelastic, anisotropic

Impact of machine learning

- Traditional continuum mechanics: simple constitutive hypotheses
  - Hookean elasticity
  - Maxwell viscoelasticity
  - Voigt viscoelasticity
  - Newtonian fluid
- Many materials of current interest deviate from above hypotheses
  - Challenge: how to approximate $\boldsymbol{f}(\boldsymbol{u})$ (high-dimensional approximation)
  - Empirical observations: deep neural networks are good approximations

$$\boldsymbol{f}(\boldsymbol{u}) \cong \boldsymbol{g}_1(\boldsymbol{g}_2(\ldots \boldsymbol{g}_L(\boldsymbol{u})))$$

Deformation gradient	$\boldsymbol{x} = \boldsymbol{a} + \boldsymbol{u}$	$\mathrm{d}\boldsymbol{x} = \mathbf{F}\,\mathrm{d}\boldsymbol{a}$	$\mathbf{F} = \frac{\partial x}{\partial \boldsymbol{a}}$
Left Cauchy-Green	$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{x}^T \mathrm{d}\boldsymbol{x}$	$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{a}^T \mathbf{F}^T \mathbf{F}\,\mathrm{d}\boldsymbol{a}$	$\mathbf{C} = \mathbf{F}^T \mathbf{F}$
Right Cauchy-Green	$\mathrm{d}S^2 = \mathrm{d}\boldsymbol{a}^T \mathrm{d}\boldsymbol{a}$	$\mathrm{d}S^2 = \mathrm{d}\boldsymbol{x}^T \mathbf{F}^{-T} \mathbf{F}^{-1} \mathrm{d}\boldsymbol{x}$	$\mathbf{B} = \mathbf{F}\,\mathbf{F}^T$
			$\mathbf{B}^{-1} = \mathbf{F}^{-T}\,\mathbf{F}^{-1}$
Strains:	Almasi ($\mathrm{d}\boldsymbol{x}$)	Green-Lagrange ($\mathrm{d}\boldsymbol{a}$)	True ($\mathrm{d}\boldsymbol{x}$)
	$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$	$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$	$\boldsymbol{\epsilon} = -\frac{1}{2} \ln \mathbf{B}^{-1}$
		$\mathbf{E} \cong \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{a}} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{a}}^T\right)$	
SVD	$\mathbf{F} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$	$\mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbf{I}$	$\boldsymbol{\Sigma} = \mathrm{diag}(\boldsymbol{s})$
		$\mathbf{V}\mathbf{V}^T = \mathbf{V}^T\mathbf{V} = \mathbf{I}$	
Polar decomposition	$\mathbf{F} = \mathbf{R}\mathbf{S}$	$\mathbf{R} = \mathbf{U}\mathbf{V}^T$	$\mathbf{S} = \mathbf{V}\boldsymbol{\Sigma}\mathbf{V}^T$
	$\mathbf{F} = \mathbf{T}\mathbf{R}$	$\mathbf{T} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{U}^T$	$\mathbf{R} = \mathbf{U}\mathbf{V}^T$
Velocity gradient	$\mathbf{L} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}$	$\dot{\mathbf{F}} = \mathbf{L}\,\mathbf{F}$	$\mathbf{L} = \dot{\mathbf{F}}\,\mathbf{F}^{-1}$
Rate of deformation	$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$		
Continuum spin	$\mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T)$		

- Behavior of physical quantity upon change of coordinates $\boldsymbol{x}' = \mathbf{T}\boldsymbol{x}$
  - invariant: $m \to m' = m$, the scalar mass
  - covariant: $\boldsymbol{g} \to \boldsymbol{g}' = \mathbf{T}\boldsymbol{g}$, $\boldsymbol{g} = g_i \boldsymbol{e}^i$, e.g., $\boldsymbol{g} = \nabla f$
  - contravariant: $\boldsymbol{v} \to \boldsymbol{v}' = \mathbf{T}^{-1}\boldsymbol{v}$, $\boldsymbol{v} = v^i \boldsymbol{e}_i$, e.g., position vector, velocity
- In $d = 3$ dimensions, $n = d^q$ numbers are needed to specify a real physical quantity associated with q directions. In addition, physical quantities must transform as indicated above when changing coordinates. Such quantities are represented mathematically by tensors.

$q \backslash \text{Obeys}\, \mathbf{T}$ transformation?	No	Yes
0	scalar $\mathbb{R}^{d^q} = \mathbb{R}$	$\mathbb{T}_0$ rank-0 tensor
1	vector $\mathbb{R}^{d^q} = \mathbb{R}^3$	$\mathbb{T}_1$ rank-1 tensor
2	dyad (matrix) $\mathbb{R}^{d^q} = \mathbb{R}^9$	$\mathbb{T}_2$ rank-2 tensor

Note: $\mathbb{R}^{d^q} \neq \mathbb{T}_q$, $\mathbb{T}_q \subset \mathbb{R}^{d^q}, \mathbb{R}^{d^q} \subsetneq \mathbb{T}_q$ (p.9-24 of Kolecki, *An Introduction to Tensors for Students of Physics and Engineering*)

Tensor calculus

- Rank k tensor notation:
  - as geometric objects $a \in \mathbb{T}_0, \boldsymbol{v} \in \mathbb{T}_1, \mathbf{S} \in \mathbb{T}_2$
  - components: $a, v_i, v_\mu, S_{ij}, S_{\alpha\beta}$, $i, j \in \{1, \ldots, d\}$, $\alpha, \beta \in \{0, 1, \ldots, d\}$
  - r covariant, s contravariant indices: $A^{j_1 j_2 \ldots j_s}_{i_1 i_2 \ldots i_r}$
- Summation convention: $a_i b^i \equiv \sum_{i=1}^d a_i b^i$, $a_\mu b^\mu \equiv \sum_{\mu=0}^d a_\mu b^\mu$