Lesson overview

- Elastic media
- Isotropic, orthotropic media

- Adopt Lagrangean description: displacements are $\boldsymbol{u}(t,\boldsymbol{a})$
- Describe deformation by linearized Green-Lagrange tensor

$$\mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{I}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{a}} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{a}}^T + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{a}}^T \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{a}}\right) \cong \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{a}} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{a}}^T\right) \equiv \boldsymbol{\epsilon} = \boldsymbol{\epsilon}^T$$

- Conservation of momentum $\partial_t(\rho \boldsymbol{v}) = -\nabla \cdot \boldsymbol{\sigma}$, $\rho = \text{const}$

$$\rho v_{i,t} = -\sigma_{ij,j}$$

- A constitutive law $\boldsymbol{\sigma}(\boldsymbol{\epsilon})$ is required. The linear approximation is known as a generalized Hooke's law

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

with $\sigma_{ij} = \sigma_{ji}$ imposed by conservation of angular momentum, and $\epsilon_{kl} = \epsilon_{lk}$.

- A tensor is isotropic if its components have the same values in all Cartesian coordinate systems, i.e., those related by translation and rotation
- Examples: δ_{ij}, ε_{ijk}
- General form of isotropic tensors:
  - of rank 0, a (all scalars)
  - of rank 1, none exist
  - of rank 2, $\lambda\delta_{ij}$
  - of rank 3, $\mu\varepsilon_{ijk}$
  - of rank 4, $\alpha\delta_{ij}\delta_{kl}+\beta\delta_{ik}\delta_{jl}+\gamma\delta_{il}\delta_{jk}$

  General form of rank-4 tensor with symmetries $C_{ijkl}=C_{jikl}$, $C_{ijkl}=C_{ijlk}$

$$C_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$

- From $\sigma_{ij} = C_{ijkl}\,\epsilon_{kl}$, $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\,\delta_{jk})$

$$\sigma_{ij} = \lambda\delta_{ij}\,\epsilon_{kk} + \mu(\epsilon_{ij} + \epsilon_{ji})$$

- Introduce rate of strain tensor

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{a}} + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{a}}^{T}\right), \dot{\epsilon}_{ij} = \epsilon_{ij,t} = \frac{1}{2}(v_{i,j} + v_{j,i})$$

- Cauchy PDE system for linear elasticity

$$\begin{aligned} \rho v_{i,t} &= -\sigma_{ij,j} \\ \sigma_{ij,t} &= \lambda\delta_{ij}\,\dot{\epsilon}_{kk} + \mu(\dot{\epsilon}_{ij} + \dot{\epsilon}_{ji}) \\ &= \lambda\delta_{ij}\,v_{k,k} + \mu(v_{i,j} + v_{j,i}) \end{aligned}$$

Analytical mechanics derivation of 1D linear elasticity equations

- Recall that mechanics can be derived from:
  - application of conservation laws, e.g. Newton's law: $(\rho \boldsymbol{v})_t = -\nabla \cdot \boldsymbol{\sigma}$
  - Euler's equations from variational calculus

$$\mathcal{L} = \mathcal{K} - \mathcal{E}, \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \left(\frac{\partial \mathcal{L}}{\partial q}\right) = f$$

- For 1D deformation, the kinetic and strain energy density are

$$\mathcal{K} = \frac{1}{2}\rho u_t^2, \mathcal{E} = \frac{1}{2}E\epsilon^2 = \frac{1}{2}E u_x^2 \Rightarrow \sigma = \frac{\partial \mathcal{E}}{\partial \epsilon} = E\epsilon\,(\text{statics})$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) = 0 \Rightarrow u_{tt} - c^2 u_{xx} = 0, c^2 = E/\rho\,(\text{dynamics})$$

In 3D

$$\mathcal{E} = \frac{1}{2} C_{klmn}\, \epsilon_{kl} \epsilon_{mn} \Rightarrow \sigma_{ij} = \frac{\partial \mathcal{E}}{\partial \epsilon_{ij}} = \frac{1}{2} C_{klmn} \left(\frac{\partial \epsilon_{kl}}{\partial \epsilon_{ij}} \epsilon_{mn} + \epsilon_{kl} \frac{\partial \epsilon_{mn}}{\partial \epsilon_{ij}} \right)$$

$$\sigma_{ij} = \frac{1}{2} C_{klmn} \left(\delta_{ik} \delta_{jl} \epsilon_{mn} + \epsilon_{kl} \delta_{im} \delta_{jn} \right) = \frac{1}{2} C_{ijmn} \epsilon_{mn} + \frac{1}{2} C_{klij} \epsilon_{kl}$$

Since $\mathcal{E} = \frac{1}{2} C_{klmn}\, \epsilon_{kl} \epsilon_{mn} = \frac{1}{2} C_{mnkl}\, \epsilon_{mn} \epsilon_{kl}$ it results that $C_{ijkl} = C_{klij}$, and

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl}\ (\text{statics})$$

From $\mathcal{K} = \frac{1}{2} \rho u_{j,t} u_{j,t}$, $\mathcal{E} = \frac{1}{8} C_{klmn} (u_{k,l} + u_{l,k})(u_{m,n} + u_{n,m})$

$$\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial u_{i,t}} \right) + \frac{\partial}{\partial a_j} \left(\frac{\partial \mathcal{L}}{\partial u_{i,j}} \right) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_{i,t}}\right)=\frac{\partial}{\partial t}\left(\frac{\partial}{\partial u_{i,t}}\frac{1}{2}\rho u_{j,t}u_{j,t}\right)=\frac{1}{2}\rho\frac{\partial}{\partial t}(\delta_{ij}u_{j,t}+u_{j,t}\delta_{ij})=\rho u_{i,tt}$$

$$\left.\frac{\partial \mathcal{L}}{\partial u_{i,j}}\right|_{,j}=\tfrac{1}{8}C_{klmn}[(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})(u_{m,n}+u_{n,m})+(u_{k,l}+u_{l,k})(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})]_{,j}$$

$$\left.\frac{\partial \mathcal{L}}{\partial u_{i,j}}\right|_{,j}=\frac{1}{4}[C_{ijkl}(u_{k,lj}+u_{l,kj})+C_{klij}(u_{k,lj}+u_{l,kj})]=\frac{1}{2}C_{ijkl}(u_{k,lj}+u_{l,kj})$$

$$C_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$

$$\left.\frac{\partial \mathcal{L}}{\partial u_{i,j}}\right|_{,j}=\lambda u_{k,ki}+\mu(u_{i,jj}+u_{j,ij})$$

$$\rho u_{i,tt}+\lambda u_{k,ki}+\mu(u_{i,jj}+u_{j,ij})=0\Leftrightarrow\rho\boldsymbol{u}_{tt}+(\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}+\mu\nabla^2\boldsymbol{u}=0$$