Lesson overview

- Hyperelastic materials
- Plastic behavior

- Compare $(\rho v_i)_t = \sigma_{ij,j}$, $v_i = u_{i,t}$ with

$$\mathcal{L} = \mathcal{K} - \mathcal{E}, \mathcal{K} = \frac{1}{2}\rho u_{j,t} u_{j,t}, \mathcal{E} = \mathcal{E}(\mathbf{F}), \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_{i,t}}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial \mathcal{L}}{\partial u_{i,j}}\right) = 0$$

  and identify

$$\sigma_{ij} = -\frac{\partial \mathcal{L}}{\partial u_{i,j}} = \frac{\partial \mathcal{E}}{\partial u_{i,j}}$$

- Materials for which the strain energy can be expressed in terms of the deformation gradient $\mathbf{F} = \frac{\partial \boldsymbol{x}}{\partial \mathbf{X}} = \boldsymbol{I} + \frac{\partial \boldsymbol{u}}{\partial \mathbf{X}}$, are said to be *hyperelastic*. Examples:
  - Linear elastic (Hookean) $\mathcal{E} = \frac{1}{2} C_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl}$, strain $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, $C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
  - Isotropic shear solid (gel) $\mathcal{E} = \mu I_2 + \frac{1}{3} A I_3 + D I_2^2$, using invariants of the Green-Lagrange strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$, $I_2 = E_{ij}E_{ji}$, $I_3 = E_{ij}E_{jk}E_{ki}$

- From definition of deformation gradient $\mathbf{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \mathbf{I} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}$ obtain

$$\mathbf{F}_t = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{X}} = \nabla \cdot (\boldsymbol{v} \otimes \mathbf{I})$$

- Combined with $\boldsymbol{v}_t = \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma}(\mathbf{F})$, a conservative formulation is obtained

$$\mathbf{q}_t + \nabla \cdot \mathbf{f}(\mathbf{q}) = 0, \mathbf{q} = \begin{pmatrix} \boldsymbol{v} \\ \mathbf{F} \end{pmatrix}, \mathbf{f}(\mathbf{q}) = \begin{pmatrix} \boldsymbol{\sigma}(\mathbf{F}) \\ \boldsymbol{v} \otimes \mathbf{I} \end{pmatrix}$$

  where all material constants within $\boldsymbol{\sigma}(\mathbf{F})$ are assumed to be scaled by the density, and the material is considered incompressible

- The linearized version of the conservative system is

$$\mathbf{q}_t + \mathbf{A} \cdot \nabla \mathbf{q} = 0, \mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}}$$

- Respecting the principle of causality requires that the eigenproblem $\mathbf{A}\mathbf{R}=\mathbf{R}\boldsymbol{\Lambda}$ admits an invertible eigenbasis $\mathbf{R}$, and the PDE system is hyperbolic

Plastic deformation

- Multiplicative decomposition of deformation gradient into elastic/plastic

$$\mathbf{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \mathbf{F}^e \mathbf{F}^p, \mathbf{F}^e = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{p}}, \mathbf{F}^p = \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{X}}, \mathrm{d}\boldsymbol{x} = \mathbf{F}\,\mathrm{d}\boldsymbol{X}$$