Overview

- Symplectic dynamical system integration: Verlet's method
- Hookean springs (linear elasticity)
- Fermi-Ulam-Past problem (non-linear elaticity, soliton solutions)
- Maxwell, Voigt elements (linear viscoelasticity)

- n point masses, at equilibrium positions $x_i = i\,h$, $u_i(t)$ is displacement from equilibrium position, fixed ends: $u_0 = u_{n+1} = 0$, length of array: $a = (n+1)h$
- Linear elasticity

$$m\,\ddot{u}_i = k(u_{i+1} - 2u_i + u_{i-1}) = f_i = -\nabla_i V(\boldsymbol{u})$$

- For linear elasticity the potential is

$$V(\boldsymbol{u}) = \frac{k}{2}\sum_{j=1}^{n+1} (u_j - u_{j-1})^2$$

- Force on point mass i

$$f_i = \frac{\partial}{\partial u_i}\frac{k}{2}\sum_{j=1}^{n+1} (u_j - u_{j-1})^2 = k[(u_i - u_{i-1}) - (u_{i+1} - u_i)]$$

Symplectic integrators: Verlet scheme

- Symplectic integrator: energy conserving $E = \|\dot{\boldsymbol{u}}\| + \|\boldsymbol{u}\| = \text{constant}$
- Verlet integration: centered finite difference of time derivative, time step δ

$$a_i^n = \ddot{u}_i(t^n) = \ddot{u}_i^n \cong \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\delta^2} \Rightarrow$$

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + \delta^2 a_i^n$$

- Verlet algorithm = leapfrog scheme, stable on the slit $[-i, i]$ in complex plane
- Consider initial conditions, displacement from equilibrium, no initial velocity

$$u_i(0) = u(0, x_i) = \sum_{k=1}^{K} a_k \sin\left(\frac{2\pi k x_i}{L}\right) = u_i(\delta)$$