Wave propagation in hyperelastic media

Sorin Mitran (mitran@unc.edu)
Department of Mathematics, University of North Carolina at Chapel Hill
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Hyperelastic media allow derivation of a constitutive stress-strain relationship from the strain energy density. This offers the opportunity for automated generation of the source code for a solver for such media using the wave propagation approach of LeVeque, a variant of the finite volume method based upon identification of the eigenmodes of a hyperbolic system of partial differential equations. Momentum conservation for hyperelastic media can be formulated as

$(\rho\, v)_t = \mathrm{div}\, \sigma(F)$

$F_t = \mathrm{div}\, (v \otimes \delta)$

with ρ - mass density, v - velocity, σ Cauchy stress, F - deformation gradient, $v \otimes \delta$ - exterior product of velocity with identity matrix. The above system can be rewritten as

$q_t + \mathrm{div}\, f(q) = 0$

with the local linearization

$q_t + A \cdot \mathrm{div}(q) = 0.$

The eigenmodes of the flux Jacobian A represent the displacement waves of the medium. This notebook defines finite strain tensors, computes the eigendecomposition $A\,R = R\,\Lambda$, and generates Fortran source code to solve the linearized Riemann problem Rδc = δq. The source code is included by the problem module of BEARCLAW to carry out a numerical solution.

Notation

Tensors are generally denoted by double strike Latin letters such as: $\mathbb{x}$, $\mathbb{F}$, with keyboard shortcuts ␛dsx␛, ␛dsF␛. The scalar components are denoted by the indexed, corresponding normal font Latin letter, $\mathbb{x} = \{x_1,\ x_2,\ x_3\}$. Tensors with standard notation by Greek letters are denoted by the corresponding script letter $\mathcal{e}$, with keyboard shortcut ␛dse␛, and scalar components denoted by the indexed Greek letters, $\mathcal{e} = \{\{\epsilon_{11},\ \epsilon_{12},\ \epsilon_{13}\},\{\epsilon_{21},\ \epsilon_{22},\ \epsilon_{23}\},\{\epsilon_{31},\ \epsilon_{32},\ \epsilon_{33}\}\}$. When the scalar components of the tensor are not used, a tensor can also be denoted by a normal face Greek letter.

Define number of space dimensions

In[2]:=
```
d = 3
```
Out[2]=
```
3
```

Define identity matrix, Dirac delta

In[3]:=
```
𝕀 = IdentityMatrix[d]; δ = 𝕀; MatrixForm[δ]
```
Out[3]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Define some utility functions for Einstein summation notation

In[4]:=
```
idx2[i_, j_] := ToString[i] <> ToString[j];
idx3[i_, j_, k_] := ToString[i] <> ToString[j] <> ToString[k];
idx4[i_, j_, k_, l_] := ToString[i] <> ToString[j] <> ToString[k] <> ToString[l];
```

Deformation measures

Displacement

Define reference, current coordinates as independent variables

In[7]:=
```
𝕏 = Table[X_i, {i, d}]; 𝕩 = Table[x_i, {i, d}]; TableForm[{𝕏, 𝕩}]
```
Out[7]//TableForm=

X_1	X_2	X_3
x_1	x_2	x_3

Define current coordinates as dependent variables

In[8]:=
```
𝕩𝕩 = Table[𝕏[[i]] + u_i[t, 𝕏], {i, d}]; TableForm[𝕩𝕩]
```
Out[8]//TableForm=

$X_1 + u_1[t, \{X_1, X_2, X_3\}]$
$X_2 + u_2[t, \{X_1, X_2, X_3\}]$
$X_3 + u_3[t, \{X_1, X_2, X_3\}]$

Define displacements as independent variables

In[9]:=
```
𝕦 = Table[u_i, {i, d}]; TableForm[{𝕦}]
```
Out[9]//TableForm=

u_1	u_2	u_3

Define displacements as dependent variables w.r.t. reference coordinates

In[10]:=
```
uX = Table[u_i[t, X], {i, d}]; TableForm[{uX}]
```

Out[10]//TableForm=

$u_1[t, \{X_1, X_2, X_3\}]$	$u_2[t, \{X_1, X_2, X_3\}]$	$u_3[t, \{X_1, X_2, X_3\}]$

Deformation tensors

Define deformation gradient in terms of displacements taken as independent variables

In[11]:=
```
F = δ + Table[u_{i,j}, {i, d}, {j, d}]; TableForm[F]
```

Out[11]//TableForm=

$1+u_{1,1}$	$u_{1,2}$	$u_{1,3}$
$u_{2,1}$	$1+u_{2,2}$	$u_{2,3}$
$u_{3,1}$	$u_{3,2}$	$1+u_{3,3}$

Define deformation gradient with displacements taken as variables that depend on reference configuration

In[12]:=
```
FX = δ + Table[D[uX[[i]], X[[j]]], {i, d}, {j, d}]; TableForm[FX]
```

Out[12]//TableForm=

$1+u_1^{(0,\{1,0,0\})}[t, \{X_1, X_2, X_3\}]$	$u_1^{(0,\{0,1,0\})}[t, \{X_1, X_2, X_3\}]$	$u_1^{(0,\{0,0,1\})}[t, \{X_1, X_2, X_3\}]$
$u_2^{(0,\{1,0,0\})}[t, \{X_1, X_2, X_3\}]$	$1+u_2^{(0,\{0,1,0\})}[t, \{X_1, X_2, X_3\}]$	$u_2^{(0,\{0,0,1\})}[t, \{X_1, X_2, X_3\}]$
$u_3^{(0,\{1,0,0\})}[t, \{X_1, X_2, X_3\}]$	$u_3^{(0,\{0,1,0\})}[t, \{X_1, X_2, X_3\}]$	$1+u_3^{(0,\{0,0,1\})}[t, \{X_1, X_2, X_3$

Define left Cauchy-Green deformation tensor in terms of deformation gradient

In[13]:=
```
B = F.Transpose[F]; TableForm[B]
```

Out[13]//TableForm=

$(1+u_{1,1})^2+u_{1,2}^2+u_{1,3}^2$	$(1+u_{1,1})\,u_{2,1}+u_{1,2}\,(1+u_{2,2})+u_{1,3}\,u_{2,3}$	$(1+u_{1,1})\,u_{3,}$
$(1+u_{1,1})\,u_{2,1}+u_{1,2}\,(1+u_{2,2})+u_{1,3}\,u_{2,3}$	$u_{2,1}^2+(1+u_{2,2})^2+u_{2,3}^2$	$u_{2,1}\,u_{3,1}+(1$
$(1+u_{1,1})\,u_{3,1}+u_{1,2}\,u_{3,2}+u_{1,3}\,(1+u_{3,3})$	$u_{2,1}\,u_{3,1}+(1+u_{2,2})\,u_{3,2}+u_{2,3}\,(1+u_{3,3})$	$u_{3,1}^2+u_{3,2}^2+$

Define right Cauchy-Green deformation tensor in terms of displacement functions

In[14]:=
```
BX = FX.Transpose[FX]; TableForm[BX]
```

Out[14]//TableForm=

$u_1^{(0,\{0,0,1\})}[t, \{X_1, X_2, X_3\}]^2 + u_1^{(0,\{0,1,0\})}[t, \{X_1, X_2, X_3\}]^2 + (1 + u_1^{(0,\{1,0,0\})}[t, \{X_1, X_2, X_3\}])^2$

$u_1^{(0,\{0,0,1\})}[t, \{X_1, X_2, X_3\}]\,u_2^{(0,\{0,0,1\})}[t, \{X_1, X_2, X_3\}] + u_1^{(0,\{0,1,0\})}[t, \{X_1, X_2, X_3\}]\,(1 + u_2^{(0,\{0,1,0\}}$

$u_1^{(0,\{0,0,1\})}[t, \{X_1, X_2, X_3\}]\,(1 + u_3^{(0,\{0,0,1\})}[t, \{X_1, X_2, X_3\}]) + u_1^{(0,\{0,1,0\})}[t, \{X_1, X_2, X_3\}]\,u_3^{(0,\{0,1,0}$

Define right Cauchy-Green deformation tensor in terms of deformation gradient

In[15]:=
```
ℂ = Transpose[𝔽].𝔽; TableForm[ℂ]
```

Out[15]//TableForm=

$(1+u_{1,1})^2+u_{2,1}^2+u_{3,1}^2$	$(1+u_{1,1})\,u_{1,2}+u_{2,1}\,(1+u_{2,2})+u_{3,1}\,u_{3,2}$	$(1+u_{1,1})\,u_{1,}$
$(1+u_{1,1})\,u_{1,2}+u_{2,1}\,(1+u_{2,2})+u_{3,1}\,u_{3,2}$	$u_{1,2}^2+(1+u_{2,2})^2+u_{3,2}^2$	$u_{1,2}\,u_{1,3}+(1$
$(1+u_{1,1})\,u_{1,3}+u_{2,1}\,u_{2,3}+u_{3,1}\,(1+u_{3,3})$	$u_{1,2}\,u_{1,3}+(1+u_{2,2})\,u_{2,3}+u_{3,2}\,(1+u_{3,3})$	$u_{1,3}^2+u_{2,3}^2+$

Define right Cauchy-Green deformation tensor in terms of displacement functions

In[16]:=
```
ℂ𝕏 = Transpose[𝔽𝕏].𝔽𝕏; TableForm[ℂ𝕏]
```

Out[16]//TableForm=

$(1+u_1{}^{(0,\{1,0,0\})}[t,\{X_1,X_2,X_3\}])^2+u_2{}^{(0,\{1,0,0\})}[t,\{X_1,X_2,X_3\}]^2+u_3{}^{(0,\{1,0,0\})}[t,\{X_1,X_2,X_3\}]^2$

$u_1{}^{(0,\{0,1,0\})}[t,\{X_1,X_2,X_3\}]\,(1+u_1{}^{(0,\{1,0,0\})}[t,\{X_1,X_2,X_3\}])+(1+u_2{}^{(0,\{0,1,0\})}[t,\{X_1,X_2,X_3\}])\,u_2{}^{(0,\{}$

$u_1{}^{(0,\{0,0,1\})}[t,\{X_1,X_2,X_3\}]\,(1+u_1{}^{(0,\{1,0,0\})}[t,\{X_1,X_2,X_3\}])+u_2{}^{(0,\{0,0,1\})}[t,\{X_1,X_2,X_3\}]\,u_2{}^{(0,\{1,0,0}$

Strain

Linear engineering strain

In terms of deformation gradient

In[17]:=
```
e = Table[1/2 (u_{i,j} + u_{j,i}), {i, d}, {j, d}]; TableForm[e]
```

Out[17]//TableForm=

$u_{1,1}$	$\frac{1}{2}(u_{1,2}+u_{2,1})$	$\frac{1}{2}(u_{1,3}+u_{3,1})$
$\frac{1}{2}(u_{1,2}+u_{2,1})$	$u_{2,2}$	$\frac{1}{2}(u_{2,3}+u_{3,2})$
$\frac{1}{2}(u_{1,3}+u_{3,1})$	$\frac{1}{2}(u_{2,3}+u_{3,2})$	$u_{3,3}$

In terms of displacement functions

In[18]:=
```
e𝕏 = Table[1/2 (D[u𝕏[[i]], X_j] + D[u𝕏[[j]], X_i]), {i, d}, {j, d}]; TableForm[e𝕏]
```

Out[18]//TableForm=

$u_1{}^{(0,\{1,0,0\})}[t,\{X_1,X_2,X_3\}]$	$\frac{1}{2}(u_1{}^{(0,\{0,1,0\})}[t,\{X_1,X_2,X_3\}]+u_2{}^{(}$
$\frac{1}{2}(u_1{}^{(0,\{0,1,0\})}[t,\{X_1,X_2,X_3\}]+u_2{}^{(0,\{1,0,0\})}[t,\{X_1,X_2,X_3\}])$	$u_2{}^{(0,\{0,1,0\})}[t,\{X_1,X_2,X_3\}]$
$\frac{1}{2}(u_1{}^{(0,\{0,0,1\})}[t,\{X_1,X_2,X_3\}]+u_3{}^{(0,\{1,0,0\})}[t,\{X_1,X_2,X_3\}])$	$\frac{1}{2}(u_2{}^{(0,\{0,0,1\})}[t,\{X_1,X_2,X_3\}]+u_3{}^{(}$

Green-Lagrange (Green-St. Venant) strain

In terms of deformation gradient

In[19]:=
```
𝔼 = Expand[1/2 (ℂ − 𝕀)]; TableForm[𝔼]
```

Out[19]//TableForm=

$u_{1,1} + \frac{u_{1,1}^2}{2} + \frac{u_{2,1}^2}{2} + \frac{u_{3,1}^2}{2}$	$\frac{u_{1,2}}{2} + \frac{1}{2} u_{1,1} u_{1,2} + \frac{u_{2,1}}{2} + \frac{1}{2} u_{2,1} u_{2,2} + \frac{1}{2} u_{3,1} u_{3,}$
$\frac{u_{1,2}}{2} + \frac{1}{2} u_{1,1} u_{1,2} + \frac{u_{2,1}}{2} + \frac{1}{2} u_{2,1} u_{2,2} + \frac{1}{2} u_{3,1} u_{3,2}$	$\frac{u_{1,2}^2}{2} + u_{2,2} + \frac{u_{2,2}^2}{2} + \frac{u_{3,2}^2}{2}$
$\frac{u_{1,3}}{2} + \frac{1}{2} u_{1,1} u_{1,3} + \frac{1}{2} u_{2,1} u_{2,3} + \frac{u_{3,1}}{2} + \frac{1}{2} u_{3,1} u_{3,3}$	$\frac{1}{2} u_{1,2} u_{1,3} + \frac{u_{2,3}}{2} + \frac{1}{2} u_{2,2} u_{2,3} + \frac{u_{3,2}}{2} + \frac{1}{2} u_{3,2} u_{3,}$

In terms of displacement functions

In[20]:=
```
𝔼𝕏 = Expand[1/2 (ℂ𝕏 − 𝕀)]; TableForm[𝔼𝕏]
```

Out[20]//TableForm=

$u_1^{(0,\{1,0,0\})}[t, \{X_1, X_2, X_3\}] + \frac{1}{2} u_1^{(0,\{1,0,0\})}[t, \{X_1, X_2, X_3\}]^2 + \frac{1}{2} u_2^{(0,\{1,0,0\})}[t, \{X_1, X_2, X_3\}]^2 + \frac{1}{2} u_3^{(}$

$\frac{1}{2} u_1^{(0,\{0,1,0\})}[t, \{X_1, X_2, X_3\}] + \frac{1}{2} u_1^{(0,\{0,1,0\})}[t, \{X_1, X_2, X_3\}] u_1^{(0,\{1,0,0\})}[t, \{X_1, X_2, X_3\}] + \frac{1}{2} u_2^{(0,\{1,}$

$\frac{1}{2} u_1^{(0,\{0,0,1\})}[t, \{X_1, X_2, X_3\}] + \frac{1}{2} u_1^{(0,\{0,0,1\})}[t, \{X_1, X_2, X_3\}] u_1^{(0,\{1,0,0\})}[t, \{X_1, X_2, X_3\}] + \frac{1}{2} u_2^{(0,\{0,}$

Constitutive relations

Linear elastic (Hooke)

In[21]:=
```
ℂLE = Table[λ δ[[i, j]] × δ[[k, l]] + μ (δ[[i, k]] × δ[[j, l]] + δ[[i, l]] × δ[[j, k]]),
  {i, d}, {j, d}, {k, d}, {l, d}]; TableForm[ℂLE]
```

Out[21]//TableForm=

$\lambda + 2\mu$	0	0	0	μ	0	0	0	μ
0	λ	0	μ	0	0	0	0	0
0	0	λ	0	0	0	μ	0	0
0	μ	0	λ	0	0	0	0	0
μ	0	0	0	$\lambda + 2\mu$	0	0	0	μ
0	0	0	0	0	λ	0	μ	0
0	0	μ	0	0	0	λ	0	0
0	0	0	0	0	μ	0	λ	0
μ	0	0	0	μ	0	0	0	$\lambda + 2\mu$

Strain energy

Define potential for various types of elastic materials

Linear elastic (Hooke)

In[22]:=
```
ℰ_LE = 1/2 Sum[ℂ_LE[[i, j, k, l]] ϵ[[i, j]] × ϵ[[k, l]], {i, d}, {j, d}, {k, d}, {l, d}]
```

Out[22]=

$$\frac{1}{2}\left((\lambda+2\,\mu)\,u_{1,1}^2+\mu\,(u_{1,2}+u_{2,1})^2+2\,\lambda\,u_{1,1}\,u_{2,2}+(\lambda+2\,\mu)\,u_{2,2}^2+\mu\,(u_{1,3}+u_{3,1})^2+\mu\,(u_{2,3}+u_{3,2})^2+2\,\lambda\,u_{1,1}\,u_{3,3}+2\,\lambda\,u_{2,2}\,u_{3,3}+(\lambda+2\,\mu)\,u_{3,3}^2\right)$$

In[23]:=
```
𝒰_LE = Expand[ℰ_LE]
```

Out[23]=

$$\frac{1}{2}\lambda\,u_{1,1}^2+\mu\,u_{1,1}^2+\frac{1}{2}\mu\,u_{1,2}^2+\frac{1}{2}\mu\,u_{1,3}^2+\mu\,u_{1,2}\,u_{2,1}+\frac{1}{2}\mu\,u_{2,1}^2+\lambda\,u_{1,1}\,u_{2,2}+\frac{1}{2}\lambda\,u_{2,2}^2+\mu\,u_{2,2}^2+\frac{1}{2}\mu\,u_{2,3}^2+\mu\,u_{1,3}\,u_{3,1}+\frac{1}{2}\mu\,u_{3,1}^2+\mu\,u_{2,3}\,u_{3,2}+\frac{1}{2}\mu\,u_{3,2}^2+\lambda\,u_{1,1}\,u_{3,3}+\lambda\,u_{2,2}\,u_{3,3}+\frac{1}{2}\lambda\,u_{3,3}^2+\mu\,u_{3,3}^2$$

Isotropic soft solid shear modes

Define invariants of Green-Lagrange strain tensor

In[24]:=
```
I_2 = Sum[𝔼[[i, j]] × 𝔼[[j, i]], {i, d}, {j, d}]
```

Out[24]=

$$\left(u_{1,1}+\frac{u_{1,1}^2}{2}+\frac{u_{2,1}^2}{2}+\frac{u_{3,1}^2}{2}\right)^2+2\left(\frac{u_{1,2}}{2}+\frac{1}{2}u_{1,1}\,u_{1,2}+\frac{u_{2,1}}{2}+\frac{1}{2}u_{2,1}\,u_{2,2}+\frac{1}{2}u_{3,1}\,u_{3,2}\right)^2+\left(\frac{u_{1,2}^2}{2}+u_{2,2}+\frac{u_{2,2}^2}{2}+\frac{u_{3,2}^2}{2}\right)^2+2\left(\frac{u_{1,3}}{2}+\frac{1}{2}u_{1,1}\,u_{1,3}+\frac{1}{2}u_{2,1}\,u_{2,3}+\frac{u_{3,1}}{2}+\frac{1}{2}u_{3,1}\,u_{3,3}\right)^2+2\left(\frac{1}{2}u_{1,2}\,u_{1,3}+\frac{u_{2,3}}{2}+\frac{1}{2}u_{2,2}\,u_{2,3}+\frac{u_{3,2}}{2}+\frac{1}{2}u_{3,2}\,u_{3,3}\right)^2+\left(\frac{u_{1,3}^2}{2}+\frac{u_{2,3}^2}{2}+u_{3,3}+\frac{u_{3,3}^2}{2}\right)^2$$

In[25]:=

```
𝒰₂ = Expand[I₂]
```

Out[25]=

$$
\begin{aligned}
& u_{1,1}^2 + u_{1,1}^3 + \frac{u_{1,1}^4}{4} + \frac{u_{1,2}^2}{2} + u_{1,1}\, u_{1,2}^2 + \frac{1}{2} u_{1,1}^2\, u_{1,2}^2 + \frac{u_{1,2}^4}{4} + \frac{u_{1,3}^2}{2} + u_{1,1}\, u_{1,3}^2 + \frac{1}{2} u_{1,1}^2\, u_{1,3}^2 + \\
& \frac{1}{2} u_{1,2}^2\, u_{1,3}^2 + \frac{u_{1,3}^4}{4} + u_{1,2}\, u_{2,1} + u_{1,1}\, u_{1,2}\, u_{2,1} + \frac{u_{2,1}^2}{2} + u_{1,1}\, u_{2,1}^2 + \frac{1}{2} u_{1,1}^2\, u_{2,1}^2 + \\
& \frac{u_{2,1}^4}{4} + u_{1,2}^2\, u_{2,2} + u_{1,2}\, u_{2,1}\, u_{2,2} + u_{1,1}\, u_{1,2}\, u_{2,1}\, u_{2,2} + u_{2,1}^2\, u_{2,2} + u_{2,2}^2 + \frac{1}{2} u_{1,2}^2\, u_{2,2}^2 + \\
& \frac{1}{2} u_{2,1}^2\, u_{2,2}^2 + u_{2,2}^3 + \frac{u_{2,2}^4}{4} + u_{1,2}\, u_{1,3}\, u_{2,3} + u_{1,3}\, u_{2,1}\, u_{2,3} + u_{1,1}\, u_{1,3}\, u_{2,1}\, u_{2,3} + \\
& u_{1,2}\, u_{1,3}\, u_{2,2}\, u_{2,3} + \frac{u_{2,3}^2}{2} + \frac{1}{2} u_{1,3}^2\, u_{2,3}^2 + \frac{1}{2} u_{2,1}^2\, u_{2,3}^2 + u_{2,2}\, u_{2,3}^2 + \frac{1}{2} u_{2,2}^2\, u_{2,3}^2 + \frac{u_{2,3}^4}{4} + \\
& u_{1,3}\, u_{3,1} + u_{1,1}\, u_{1,3}\, u_{3,1} + u_{2,1}\, u_{2,3}\, u_{3,1} + \frac{u_{3,1}^2}{2} + u_{1,1}\, u_{3,1}^2 + \frac{1}{2} u_{1,1}^2\, u_{3,1}^2 + \frac{1}{2} u_{2,1}^2\, u_{3,1}^2 + \\
& \frac{u_{3,1}^4}{4} + u_{1,2}\, u_{1,3}\, u_{3,2} + u_{2,3}\, u_{3,2} + u_{2,2}\, u_{2,3}\, u_{3,2} + u_{1,2}\, u_{3,1}\, u_{3,2} + u_{1,1}\, u_{1,2}\, u_{3,1}\, u_{3,2} + \\
& u_{2,1}\, u_{3,1}\, u_{3,2} + u_{2,1}\, u_{2,2}\, u_{3,1}\, u_{3,2} + \frac{u_{3,2}^2}{2} + \frac{1}{2} u_{1,2}^2\, u_{3,2}^2 + u_{2,2}\, u_{3,2}^2 + \frac{1}{2} u_{2,2}^2\, u_{3,2}^2 + \\
& \frac{1}{2} u_{3,1}^2\, u_{3,2}^2 + \frac{u_{3,2}^4}{4} + u_{1,3}^2\, u_{3,3} + u_{2,3}^2\, u_{3,3} + u_{1,3}\, u_{3,1}\, u_{3,3} + u_{1,1}\, u_{1,3}\, u_{3,1}\, u_{3,3} + \\
& u_{2,1}\, u_{2,3}\, u_{3,1}\, u_{3,3} + u_{3,1}^2\, u_{3,3} + u_{1,2}\, u_{1,3}\, u_{3,2}\, u_{3,3} + u_{2,3}\, u_{3,2}\, u_{3,3} + u_{2,2}\, u_{2,3}\, u_{3,2}\, u_{3,3} + \\
& u_{3,2}^2\, u_{3,3} + u_{3,3}^2 + \frac{1}{2} u_{1,3}^2\, u_{3,3}^2 + \frac{1}{2} u_{2,3}^2\, u_{3,3}^2 + \frac{1}{2} u_{3,1}^2\, u_{3,3}^2 + \frac{1}{2} u_{3,2}^2\, u_{3,3}^2 + u_{3,3}^3 + \frac{u_{3,3}^4}{4}
\end{aligned}
$$

In[26]:=

```
I3 = Sum[𝔼[[i, j]] × 𝔼[[j, k]] × 𝔼[[k, i]], {i, d}, {j, d}, {k, d}]
```

Out[26]=

$$\left(u_{1,1}+\frac{u_{1,1}^2}{2}+\frac{u_{2,1}^2}{2}+\frac{u_{3,1}^2}{2}\right)^3+$$
$$3\left(u_{1,1}+\frac{u_{1,1}^2}{2}+\frac{u_{2,1}^2}{2}+\frac{u_{3,1}^2}{2}\right)\left(\frac{u_{1,2}}{2}+\frac{1}{2}u_{1,1}u_{1,2}+\frac{u_{2,1}}{2}+\frac{1}{2}u_{2,1}u_{2,2}+\frac{1}{2}u_{3,1}u_{3,2}\right)^2+$$
$$3\left(\frac{u_{1,2}}{2}+\frac{1}{2}u_{1,1}u_{1,2}+\frac{u_{2,1}}{2}+\frac{1}{2}u_{2,1}u_{2,2}+\frac{1}{2}u_{3,1}u_{3,2}\right)^2\left(\frac{u_{1,2}^2}{2}+u_{2,2}+\frac{u_{2,2}^2}{2}+\frac{u_{3,2}^2}{2}\right)+$$
$$\left(\frac{u_{1,2}^2}{2}+u_{2,2}+\frac{u_{2,2}^2}{2}+\frac{u_{3,2}^2}{2}\right)^3+$$
$$3\left(u_{1,1}+\frac{u_{1,1}^2}{2}+\frac{u_{2,1}^2}{2}+\frac{u_{3,1}^2}{2}\right)\left(\frac{u_{1,3}}{2}+\frac{1}{2}u_{1,1}u_{1,3}+\frac{1}{2}u_{2,1}u_{2,3}+\frac{u_{3,1}}{2}+\frac{1}{2}u_{3,1}u_{3,3}\right)^2+$$
$$6\left(\frac{u_{1,2}}{2}+\frac{1}{2}u_{1,1}u_{1,2}+\frac{u_{2,1}}{2}+\frac{1}{2}u_{2,1}u_{2,2}+\frac{1}{2}u_{3,1}u_{3,2}\right)$$
$$\left(\frac{u_{1,3}}{2}+\frac{1}{2}u_{1,1}u_{1,3}+\frac{1}{2}u_{2,1}u_{2,3}+\frac{u_{3,1}}{2}+\frac{1}{2}u_{3,1}u_{3,3}\right)$$
$$\left(\frac{1}{2}u_{1,2}u_{1,3}+\frac{u_{2,3}}{2}+\frac{1}{2}u_{2,2}u_{2,3}+\frac{u_{3,2}}{2}+\frac{1}{2}u_{3,2}u_{3,3}\right)+$$
$$3\left(\frac{u_{1,2}^2}{2}+u_{2,2}+\frac{u_{2,2}^2}{2}+\frac{u_{3,2}^2}{2}\right)\left(\frac{1}{2}u_{1,2}u_{1,3}+\frac{u_{2,3}}{2}+\frac{1}{2}u_{2,2}u_{2,3}+\frac{u_{3,2}}{2}+\frac{1}{2}u_{3,2}u_{3,3}\right)^2+$$
$$3\left(\frac{u_{1,3}}{2}+\frac{1}{2}u_{1,1}u_{1,3}+\frac{1}{2}u_{2,1}u_{2,3}+\frac{u_{3,1}}{2}+\frac{1}{2}u_{3,1}u_{3,3}\right)^2\left(\frac{u_{1,3}^2}{2}+\frac{u_{2,3}^2}{2}+u_{3,3}+\frac{u_{3,3}^2}{2}\right)+$$
$$3\left(\frac{1}{2}u_{1,2}u_{1,3}+\frac{u_{2,3}}{2}+\frac{1}{2}u_{2,2}u_{2,3}+\frac{u_{3,2}}{2}+\frac{1}{2}u_{3,2}u_{3,3}\right)^2\left(\frac{u_{1,3}^2}{2}+\frac{u_{2,3}^2}{2}+u_{3,3}+\frac{u_{3,3}^2}{2}\right)+$$
$$\left(\frac{u_{1,3}^2}{2}+\frac{u_{2,3}^2}{2}+u_{3,3}+\frac{u_{3,3}^2}{2}\right)^3$$

In[27]:=

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𝒰3 = Expand[I3]
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Out[27]=

$$u_{1,1}^3+\frac{3u_{1,1}^4}{2}+\frac{3u_{1,1}^5}{4}+\frac{u_{1,1}^6}{8}+\frac{3}{4}u_{1,1}u_{1,2}^2+\frac{15}{8}u_{1,1}^2u_{1,2}^2+\frac{3}{2}u_{1,1}^3u_{1,2}^2+\frac{3}{8}u_{1,1}^4u_{1,2}^2+\frac{3u_{1,2}^4}{8}+$$
$$\frac{3}{4}u_{1,1}u_{1,2}^4+\frac{3}{8}u_{1,1}^2u_{1,2}^4+\frac{u_{1,2}^6}{8}+\frac{3}{4}u_{1,1}u_{1,3}^2+\frac{15}{8}u_{1,1}^2u_{1,3}^2+\frac{3}{2}u_{1,1}^3u_{1,3}^2+\frac{3}{8}u_{1,1}^4u_{1,3}^2+$$

$$\frac{3}{4} u_{1,2}^2 u_{1,3}^2 + \frac{3}{2} u_{1,1} u_{1,2}^2 u_{1,3}^2 + \frac{3}{4} u_{1,1}^2 u_{1,2}^2 u_{1,3}^2 + \frac{3}{8} u_{1,2}^4 u_{1,3}^2 + \frac{3 u_{1,3}^4}{8} + \frac{3}{4} u_{1,1} u_{1,3}^4 +$$

$$\frac{3}{8} u_{1,1}^2 u_{1,3}^4 + \frac{3}{8} u_{1,2}^2 u_{1,3}^4 + \frac{u_{1,3}^6}{8} + \frac{3}{2} u_{1,1} u_{1,2} u_{2,1} + \frac{9}{4} u_{1,1}^2 u_{1,2} u_{2,1} + \frac{3}{4} u_{1,1}^3 u_{1,2} u_{2,1} +$$

$$\frac{3}{4} u_{1,2}^3 u_{2,1} + \frac{3}{4} u_{1,1} u_{1,2}^3 u_{2,1} + \frac{3}{4} u_{1,2} u_{1,3}^2 u_{2,1} + \frac{3}{4} u_{1,1} u_{1,2} u_{1,3}^2 u_{2,1} + \frac{3}{4} u_{1,1} u_{2,1}^2 +$$

$$\frac{15}{8} u_{1,1}^2 u_{2,1}^2 + \frac{3}{2} u_{1,1}^3 u_{2,1}^2 + \frac{3}{8} u_{1,1}^4 u_{2,1}^2 + \frac{3}{4} u_{1,2}^2 u_{2,1}^2 + \frac{3}{4} u_{1,1} u_{1,2}^2 u_{2,1}^2 + \frac{3}{8} u_{1,1}^2 u_{1,2}^2 u_{2,1}^2 +$$

$$\frac{3}{8} u_{1,3}^2 u_{2,1}^2 + \frac{3}{4} u_{1,1} u_{1,3}^2 u_{2,1}^2 + \frac{3}{8} u_{1,1}^2 u_{1,3}^2 u_{2,1}^2 + \frac{3}{4} u_{1,2} u_{2,1}^3 + \frac{3}{4} u_{1,1} u_{1,2} u_{2,1}^3 + \frac{3 u_{2,1}^4}{8} +$$

$$\frac{3}{4} u_{1,1} u_{2,1}^4 + \frac{3}{8} u_{1,1}^2 u_{2,1}^4 + \frac{u_{2,1}^6}{8} + \frac{3}{4} u_{1,2}^2 u_{2,2} + \frac{3}{2} u_{1,1} u_{1,2}^2 u_{2,2} + \frac{3}{4} u_{1,1}^2 u_{1,2}^2 u_{2,2} +$$

$$\frac{3}{4} u_{1,2}^4 u_{2,2} + \frac{3}{4} u_{1,2}^2 u_{1,3}^2 u_{2,2} + \frac{3}{2} u_{1,2} u_{2,1} u_{2,2} + 3 u_{1,1} u_{1,2} u_{2,1} u_{2,2} + \frac{9}{4} u_{1,1}^2 u_{1,2} u_{2,1} u_{2,2} +$$

$$\frac{3}{4} u_{1,1}^3 u_{1,2} u_{2,1} u_{2,2} + \frac{3}{4} u_{1,2}^3 u_{2,1} u_{2,2} + \frac{3}{4} u_{1,1} u_{1,2}^3 u_{2,1} u_{2,2} + \frac{3}{4} u_{1,2} u_{1,3}^2 u_{2,1} u_{2,2} +$$

$$\frac{3}{4} u_{1,1} u_{1,2} u_{1,3}^2 u_{2,1} u_{2,2} + \frac{3}{4} u_{2,1}^2 u_{2,2} + \frac{3}{2} u_{1,1} u_{2,1}^2 u_{2,2} + \frac{3}{4} u_{1,1}^2 u_{2,1}^2 u_{2,2} + \frac{3}{4} u_{1,2}^2 u_{2,1}^2 u_{2,2} +$$

$$\frac{3}{4} u_{1,2} u_{2,1}^3 u_{2,2} + \frac{3}{4} u_{1,1} u_{1,2} u_{2,1}^3 u_{2,2} + \frac{3}{4} u_{2,1}^4 u_{2,2} + \frac{15}{8} u_{1,2}^2 u_{2,2}^2 + \frac{3}{4} u_{1,1} u_{1,2}^2 u_{2,2}^2 +$$

$$\frac{3}{8} u_{1,1}^2 u_{1,2}^2 u_{2,2}^2 + \frac{3}{8} u_{1,2}^4 u_{2,2}^2 + \frac{3}{8} u_{1,2}^2 u_{1,3}^2 u_{2,2}^2 + \frac{9}{4} u_{1,2} u_{2,1} u_{2,2}^2 + \frac{9}{4} u_{1,1} u_{1,2} u_{2,1} u_{2,2}^2 +$$

$$\frac{15}{8} u_{2,1}^2 u_{2,2}^2 + \frac{3}{4} u_{1,1} u_{2,1}^2 u_{2,2}^2 + \frac{3}{8} u_{1,1}^2 u_{2,1}^2 u_{2,2}^2 + \frac{3}{8} u_{1,2}^2 u_{2,1}^2 u_{2,2}^2 + \frac{3}{8} u_{2,1}^4 u_{2,2}^2 + u_{2,2}^3 +$$

$$\frac{3}{2} u_{1,2}^2 u_{2,2}^3 + \frac{3}{4} u_{1,2} u_{2,1} u_{2,2}^3 + \frac{3}{4} u_{1,1} u_{1,2} u_{2,1} u_{2,2}^3 + \frac{3}{2} u_{2,1}^2 u_{2,2}^3 + \frac{3 u_{2,2}^4}{2} + \frac{3}{8} u_{1,2}^2 u_{2,2}^4 +$$

$$\frac{3}{8} u_{2,1}^2 u_{2,2}^4 + \frac{3 u_{2,2}^5}{4} + \frac{u_{2,2}^6}{8} + \frac{3}{4} u_{1,2} u_{1,3} u_{2,3} + \frac{3}{2} u_{1,1} u_{1,2} u_{1,3} u_{2,3} + \frac{3}{4} u_{1,1}^2 u_{1,2} u_{1,3} u_{2,3} +$$

$$\frac{3}{4} u_{1,2}^3 u_{1,3} u_{2,3} + \frac{3}{4} u_{1,2} u_{1,3}^3 u_{2,3} + \frac{3}{4} u_{1,3} u_{2,1} u_{2,3} + \frac{9}{4} u_{1,1} u_{1,3} u_{2,1} u_{2,3} + \frac{9}{4} u_{1,1}^2 u_{1,3} u_{2,1} u_{2,3} +$$

$$\frac{3}{4} u_{1,1}^3 u_{1,3} u_{2,1} u_{2,3} + \frac{3}{4} u_{1,2}^2 u_{1,3} u_{2,1} u_{2,3} + \frac{3}{4} u_{1,1} u_{1,2}^2 u_{1,3} u_{2,1} u_{2,3} + \frac{3}{4} u_{1,3}^3 u_{2,1} u_{2,3} +$$

$$\frac{3}{4} u_{1,1} u_{1,3}^3 u_{2,1} u_{2,3} + \frac{3}{4} u_{1,2} u_{1,3} u_{2,1}^2 u_{2,3} + \frac{3}{4} u_{1,3} u_{2,1}^3 u_{2,3} + \frac{3}{4} u_{1,1} u_{1,3} u_{2,1}^3 u_{2,3} +$$

$$\frac{9}{4} u_{1,2} u_{1,3} u_{2,2} u_{2,3} + \frac{3}{2} u_{1,1} u_{1,2} u_{1,3} u_{2,2} u_{2,3} + \frac{3}{4} u_{1,1}^2 u_{1,2} u_{1,3} u_{2,2} u_{2,3} + \frac{3}{4} u_{1,2}^3 u_{1,3} u_{2,2} u_{2,3} +$$

$$\frac{3}{4} u_{1,2} u_{1,3}^3 u_{2,2} u_{2,3} + \frac{3}{2} u_{1,3} u_{2,1} u_{2,2} u_{2,3} + \frac{3}{2} u_{1,1} u_{1,3} u_{2,1} u_{2,2} u_{2,3} + \frac{3}{4} u_{1,2} u_{1,3} u_{2,1}^2 u_{2,2} u_{2,3} +$$

$$\frac{9}{4} u_{1,2} u_{1,3} u_{2,2}^2 u_{2,3} + \frac{3}{4} u_{1,3} u_{2,1} u_{2,2}^2 u_{2,3} + \frac{3}{4} u_{1,1} u_{1,3} u_{2,1} u_{2,2}^2 u_{2,3} + \frac{3}{4} u_{1,2} u_{1,3} u_{2,2}^3 u_{2,3} +$$

$$\frac{3}{8} u_{1,2}^2 u_{2,3}^2 + \frac{3}{4} u_{1,3}^2 u_{2,3}^2 + \frac{3}{4} u_{1,1} u_{1,3}^2 u_{2,3}^2 + \frac{3}{8} u_{1,1}^2 u_{1,3}^2 u_{2,3}^2 + \frac{3}{8} u_{1,2}^2 u_{1,3}^2 u_{2,3}^2 + \frac{3}{8} u_{1,3}^4 u_{2,3}^2 +$$

$$\frac{3}{4} u_{1,2} u_{2,1} u_{2,3}^2 + \frac{3}{4} u_{1,1} u_{1,2} u_{2,1} u_{2,3}^2 + \frac{3}{4} u_{2,1}^2 u_{2,3}^2 + \frac{3}{4} u_{1,1} u_{2,1}^2 u_{2,3}^2 + \frac{3}{8} u_{1,1}^2 u_{2,1}^2 u_{2,3}^2 +$$

$$\frac{3}{8} u_{1,3}^2 u_{2,1}^2 u_{2,3}^2 + \frac{3}{8} u_{2,1}^4 u_{2,3}^2 + \frac{3}{4} u_{2,2} u_{2,3}^2 + \frac{3}{4} u_{1,2}^2 u_{2,2} u_{2,3}^2 + \frac{3}{4} u_{1,3}^2 u_{2,2} u_{2,3}^2 +$$

$$\frac{3}{4} u_{1,2} u_{2,1} u_{2,2} u_{2,3}^2 + \frac{3}{4} u_{1,1} u_{1,2} u_{2,1} u_{2,2} u_{2,3}^2 + \frac{3}{2} u_{2,1}^2 u_{2,2} u_{2,3}^2 + \frac{15}{8} u_{2,2}^2 u_{2,3}^2 +$$

$$\frac{3}{8} u_{1,2}^2 u_{2,2}^2 u_{2,3}^2 + \frac{3}{8} u_{1,3}^2 u_{2,2}^2 u_{2,3}^2 + \frac{3}{4} u_{2,1}^2 u_{2,2}^2 u_{2,3}^2 + \frac{3}{2} u_{2,2}^3 u_{2,3}^2 + \frac{3}{8} u_{2,2}^4 u_{2,3}^2 + \frac{3}{4} u_{1,2} u_{1,3} u_{2,3}^3 +$$

$$\frac{3}{4} u_{1,3} u_{2,1} u_{2,3}^3 + \frac{3}{4} u_{1,1} u_{1,3} u_{2,1} u_{2,3}^3 + \frac{3}{4} u_{1,2} u_{1,3} u_{2,2} u_{2,3}^3 + \frac{3 u_{2,3}^4}{8} + \frac{3}{8} u_{1,3}^2 u_{2,3}^4 + \frac{3}{8} u_{2,1}^2 u_{2,3}^4 +$$

$$\frac{3}{4} u_{2,2} u_{2,3}^4 + \frac{3}{8} u_{2,2}^2 u_{2,3}^4 + \frac{u_{2,3}^6}{8} + \frac{3}{2} u_{1,1} u_{1,3} u_{3,1} + \frac{9}{4} u_{1,1}^2 u_{1,3} u_{3,1} + \frac{3}{4} u_{1,1}^3 u_{1,3} u_{3,1} +$$

$$\frac{3}{4} u_{1,2}^2 u_{1,3} u_{3,1} + \frac{3}{4} u_{1,1} u_{1,2}^2 u_{1,3} u_{3,1} + \frac{3}{4} u_{1,3}^3 u_{3,1} + \frac{3}{4} u_{1,1} u_{1,3}^3 u_{3,1} + \frac{3}{4} u_{1,2} u_{1,3} u_{2,1} u_{3,1} +$$

$$\frac{3}{4} u_{1,3} u_{2,1}^2 u_{3,1} + \frac{3}{4} u_{1,1} u_{1,3} u_{2,1}^2 u_{3,1} + \frac{3}{4} u_{1,2} u_{1,3} u_{2,1} u_{2,2} u_{3,1} + \frac{3}{4} u_{1,2} u_{2,3} u_{3,1} +$$

$$\frac{3}{4} u_{1,1} u_{1,2} u_{2,3} u_{3,1} + \frac{3}{4} u_{2,1} u_{2,3} u_{3,1} + \frac{3}{2} u_{1,1} u_{2,1} u_{2,3} u_{3,1} + \frac{3}{4} u_{1,1}^2 u_{2,1} u_{2,3} u_{3,1} +$$

$$\frac{3}{4} u_{1,3}^2 u_{2,1} u_{2,3} u_{3,1} + \frac{3}{4} u_{2,1}^3 u_{2,3} u_{3,1} + \frac{3}{4} u_{1,2} u_{2,2} u_{2,3} u_{3,1} + \frac{3}{4} u_{1,1} u_{1,2} u_{2,2} u_{2,3} u_{3,1} +$$

$$\frac{3}{2} u_{2,1} u_{2,2} u_{2,3} u_{3,1} + \frac{3}{4} u_{2,1} u_{2,2}^2 u_{2,3} u_{3,1} + \frac{3}{4} u_{1,3} u_{2,3}^2 u_{3,1} + \frac{3}{4} u_{1,1} u_{1,3} u_{2,3}^2 u_{3,1} +$$

$$\frac{3}{4} u_{2,1} u_{2,3}^3 u_{3,1} + \frac{3}{4} u_{1,1} u_{3,1}^2 + \frac{15}{8} u_{1,1}^2 u_{3,1}^2 + \frac{3}{2} u_{1,1}^3 u_{3,1}^2 + \frac{3}{8} u_{1,1}^4 u_{3,1}^2 + \frac{3}{8} u_{1,2}^2 u_{3,1}^2 +$$

$$\frac{3}{4} u_{1,1} u_{1,2}^2 u_{3,1}^2 + \frac{3}{8} u_{1,1}^2 u_{1,2}^2 u_{3,1}^2 + \frac{3}{4} u_{1,3}^2 u_{3,1}^2 + \frac{3}{4} u_{1,1} u_{1,3}^2 u_{3,1}^2 + \frac{3}{8} u_{1,1}^2 u_{1,3}^2 u_{3,1}^2 +$$

$$\frac{3}{4} u_{1,2} u_{2,1} u_{3,1}^2 + \frac{3}{4} u_{1,1} u_{1,2} u_{2,1} u_{3,1}^2 + \frac{3}{4} u_{2,1}^2 u_{3,1}^2 + \frac{3}{2} u_{1,1} u_{2,1}^2 u_{3,1}^2 + \frac{3}{4} u_{1,1}^2 u_{2,1}^2 u_{3,1}^2 +$$

$$\frac{3}{8} u_{2,1}^4 u_{3,1}^2 + \frac{3}{4} u_{1,2} u_{2,1} u_{2,2} u_{3,1}^2 + \frac{3}{4} u_{1,1} u_{1,2} u_{2,1} u_{2,2} u_{3,1}^2 + \frac{3}{4} u_{2,1}^2 u_{2,2} u_{3,1}^2 + \frac{3}{8} u_{2,1}^2 u_{2,2}^2 u_{3,1}^2 +$$

$$\frac{3}{4} u_{1,3} u_{2,1} u_{2,3} u_{3,1}^2 + \frac{3}{4} u_{1,1} u_{1,3} u_{2,1} u_{2,3} u_{3,1}^2 + \frac{3}{8} u_{2,3}^2 u_{3,1}^2 + \frac{3}{8} u_{2,1}^2 u_{2,3}^2 u_{3,1}^2 + \frac{3}{4} u_{1,3} u_{3,1}^3 +$$

$$\frac{3}{4} u_{1,1} u_{1,3} u_{3,1}^3 + \frac{3}{4} u_{2,1} u_{2,3} u_{3,1}^3 + \frac{3 u_{3,1}^4}{8} + \frac{3}{4} u_{1,1} u_{3,1}^4 + \frac{3}{8} u_{1,1}^2 u_{3,1}^4 + \frac{3}{8} u_{2,1}^2 u_{3,1}^4 + \frac{u_{3,1}^6}{8} +$$

$$\frac{3}{4} u_{1,2} u_{1,3} u_{3,2} + \frac{3}{2} u_{1,1} u_{1,2} u_{1,3} u_{3,2} + \frac{3}{4} u_{1,1}^2 u_{1,2} u_{1,3} u_{3,2} + \frac{3}{4} u_{1,2}^3 u_{1,3} u_{3,2} + \frac{3}{4} u_{1,2} u_{1,3}^3 u_{3,2} +$$

$$\frac{3}{4} u_{1,3} u_{2,1} u_{3,2} + \frac{3}{4} u_{1,1} u_{1,3} u_{2,1} u_{3,2} + \frac{3}{2} u_{1,2} u_{1,3} u_{2,2} u_{3,2} + \frac{3}{4} u_{1,3} u_{2,1} u_{2,2} u_{3,2} +$$

$$\frac{3}{4} u_{1,1} u_{1,3} u_{2,1} u_{2,2} u_{3,2} + \frac{3}{4} u_{1,2} u_{1,3} u_{2,2}^{2} u_{3,2} + \frac{3}{4} u_{1,2}^{2} u_{2,3} u_{3,2} + \frac{3}{4} u_{1,3}^{2} u_{2,3} u_{3,2} +$$

$$\frac{3}{4} u_{1,2} u_{2,1} u_{2,3} u_{3,2} + \frac{3}{4} u_{1,1} u_{1,2} u_{2,1} u_{2,3} u_{3,2} + \frac{3}{4} u_{2,1}^{2} u_{2,3} u_{3,2} + \frac{3}{2} u_{2,2} u_{2,3} u_{3,2} +$$

$$\frac{3}{4} u_{1,2}^{2} u_{2,2} u_{2,3} u_{3,2} + \frac{3}{4} u_{1,3}^{2} u_{2,2} u_{2,3} u_{3,2} + \frac{3}{4} u_{2,1}^{2} u_{2,2} u_{2,3} u_{3,2} + \frac{9}{4} u_{2,2}^{2} u_{2,3} u_{3,2} +$$

$$\frac{3}{4} u_{2,2}^{3} u_{2,3} u_{3,2} + \frac{3}{4} u_{1,2} u_{1,3} u_{2,3}^{2} u_{3,2} + \frac{3}{4} u_{2,3}^{3} u_{3,2} + \frac{3}{4} u_{2,2} u_{2,3}^{3} u_{3,2} + \frac{3}{4} u_{1,2} u_{3,1} u_{3,2} +$$

$$\frac{9}{4} u_{1,1} u_{1,2} u_{3,1} u_{3,2} + \frac{9}{4} u_{1,1}^{2} u_{1,2} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,1}^{3} u_{1,2} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,2}^{3} u_{3,1} u_{3,2} +$$

$$\frac{3}{4} u_{1,1} u_{1,2}^{3} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,2} u_{1,3}^{2} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,1} u_{1,2} u_{1,3}^{2} u_{3,1} u_{3,2} + \frac{3}{4} u_{2,1} u_{3,1} u_{3,2} +$$

$$\frac{3}{2} u_{1,1} u_{2,1} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,1}^{2} u_{2,1} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,2}^{2} u_{2,1} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,2} u_{2,1}^{2} u_{3,1} u_{3,2} +$$

$$\frac{3}{4} u_{1,1} u_{1,2} u_{2,1}^{2} u_{3,1} u_{3,2} + \frac{3}{4} u_{2,1}^{3} u_{3,1} u_{3,2} + \frac{3}{2} u_{1,2} u_{2,2} u_{3,1} u_{3,2} + \frac{3}{2} u_{1,1} u_{1,2} u_{2,2} u_{3,1} u_{3,2} +$$

$$\frac{9}{4} u_{2,1} u_{2,2} u_{3,1} u_{3,2} + \frac{3}{2} u_{1,1} u_{2,1} u_{2,2} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,1}^{2} u_{2,1} u_{2,2} u_{3,1} u_{3,2} +$$

$$\frac{3}{4} u_{1,2}^{2} u_{2,1} u_{2,2} u_{3,1} u_{3,2} + \frac{3}{4} u_{2,1}^{3} u_{2,2} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,2} u_{2,2}^{2} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,1} u_{1,2} u_{2,2}^{2} u_{3,1} u_{3,2} +$$

$$\frac{9}{4} u_{2,1} u_{2,2}^{2} u_{3,1} u_{3,2} + \frac{3}{4} u_{2,1} u_{2,2}^{3} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,3} u_{2,3} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,1} u_{1,3} u_{2,3} u_{3,1} u_{3,2} +$$

$$\frac{3}{4} u_{1,2} u_{1,3} u_{2,1} u_{2,3} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,3} u_{2,2} u_{2,3} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,1} u_{1,3} u_{2,2} u_{2,3} u_{3,1} u_{3,2} +$$

$$\frac{3}{4} u_{2,1} u_{2,3}^{2} u_{3,1} u_{3,2} + \frac{3}{4} u_{2,1} u_{2,2} u_{2,3}^{2} u_{3,1} u_{3,2} + \frac{3}{4} u_{1,2} u_{1,3} u_{3,1}^{2} u_{3,2} + \frac{3}{4} u_{2,3} u_{3,1}^{2} u_{3,2} +$$

$$\frac{3}{4} u_{2,2} u_{2,3} u_{3,1}^{2} u_{3,2} + \frac{3}{4} u_{1,2} u_{3,1}^{3} u_{3,2} + \frac{3}{4} u_{1,1} u_{1,2} u_{3,1}^{3} u_{3,2} + \frac{3}{4} u_{2,1} u_{3,1}^{3} u_{3,2} +$$

$$\frac{3}{4} u_{2,1} u_{2,2} u_{3,1}^{3} u_{3,2} + \frac{3}{4} u_{1,2}^{2} u_{3,2}^{2} + \frac{3}{4} u_{1,1} u_{1,2}^{2} u_{3,2}^{2} + \frac{3}{8} u_{1,1}^{2} u_{1,2}^{2} u_{3,2}^{2} + \frac{3}{8} u_{1,2}^{4} u_{3,2}^{2} +$$

$$\frac{3}{8} u_{1,3}^{2} u_{3,2}^{2} + \frac{3}{8} u_{1,2}^{2} u_{1,3}^{2} u_{3,2}^{2} + \frac{3}{4} u_{1,2} u_{2,1} u_{3,2}^{2} + \frac{3}{4} u_{1,1} u_{1,2} u_{2,1} u_{3,2}^{2} + \frac{3}{8} u_{2,1}^{2} u_{3,2}^{2} +$$

$$\frac{3}{4} u_{2,2} u_{3,2}^{2} + \frac{3}{2} u_{1,2}^{2} u_{2,2} u_{3,2}^{2} + \frac{3}{4} u_{1,2} u_{2,1} u_{2,2} u_{3,2}^{2} + \frac{3}{4} u_{1,1} u_{1,2} u_{2,1} u_{2,2} u_{3,2}^{2} +$$

$$\frac{3}{4} u_{2,1}^{2} u_{2,2} u_{3,2}^{2} + \frac{15}{8} u_{2,2}^{2} u_{3,2}^{2} + \frac{3}{4} u_{1,2}^{2} u_{2,2}^{2} u_{3,2}^{2} + \frac{3}{8} u_{2,1}^{2} u_{2,2}^{2} u_{3,2}^{2} + \frac{3}{2} u_{2,2}^{3} u_{3,2}^{2} +$$

$$\frac{3}{8} u_{2,2}^{4} u_{3,2}^{2} + \frac{3}{4} u_{1,2} u_{1,3} u_{2,3} u_{3,2}^{2} + \frac{3}{4} u_{1,2} u_{1,3} u_{2,2} u_{2,3} u_{3,2}^{2} + \frac{3}{4} u_{2,3}^{2} u_{3,2}^{2} + \frac{3}{4} u_{2,2} u_{2,3}^{2} u_{3,2}^{2} +$$

$$\frac{3}{8} u_{2,2}^2 u_{2,3}^2 u_{3,2}^2 + \frac{3}{4} u_{1,3} u_{3,1} u_{3,2}^2 + \frac{3}{4} u_{1,1} u_{1,3} u_{3,1} u_{3,2}^2 + \frac{3}{4} u_{2,1} u_{2,3} u_{3,1} u_{3,2}^2 + \frac{3}{4} u_{3,1}^2 u_{3,2}^2 +$$
$$\frac{3}{4} u_{1,1} u_{3,1}^2 u_{3,2}^2 + \frac{3}{8} u_{1,1}^2 u_{3,1}^2 u_{3,2}^2 + \frac{3}{8} u_{1,2}^2 u_{3,1}^2 u_{3,2}^2 + \frac{3}{8} u_{2,1}^2 u_{3,1}^2 u_{3,2}^2 + \frac{3}{4} u_{2,2} u_{3,1}^2 u_{3,2}^2 +$$
$$\frac{3}{8} u_{2,2}^2 u_{3,1}^2 u_{3,2}^2 + \frac{3}{8} u_{3,1}^4 u_{3,2}^2 + \frac{3}{4} u_{1,2} u_{1,3} u_{3,2}^3 + \frac{3}{4} u_{2,3} u_{3,2}^3 + \frac{3}{4} u_{2,2} u_{2,3} u_{3,2}^3 +$$
$$\frac{3}{4} u_{1,2} u_{3,1} u_{3,2}^3 + \frac{3}{4} u_{1,1} u_{1,2} u_{3,1} u_{3,2}^3 + \frac{3}{4} u_{2,1} u_{3,1} u_{3,2}^3 + \frac{3}{4} u_{2,1} u_{2,2} u_{3,1} u_{3,2}^3 + \frac{3 u_{3,2}^4}{8} +$$
$$\frac{3}{8} u_{1,2}^2 u_{3,2}^4 + \frac{3}{4} u_{2,2} u_{3,2}^4 + \frac{3}{8} u_{2,2}^2 u_{3,2}^4 + \frac{3}{8} u_{3,1}^2 u_{3,2}^4 + \frac{u_{3,2}^6}{8} + \frac{3}{4} u_{1,3}^2 u_{3,3} + \frac{3}{2} u_{1,1} u_{1,3}^2 u_{3,3} +$$
$$\frac{3}{4} u_{1,1}^2 u_{1,3}^2 u_{3,3} + \frac{3}{4} u_{1,2}^2 u_{1,3}^2 u_{3,3} + \frac{3}{4} u_{1,3}^4 u_{3,3} + \frac{3}{2} u_{1,2} u_{1,3} u_{2,3} u_{3,3} + \frac{3}{2} u_{1,3} u_{2,1} u_{2,3} u_{3,3} +$$
$$\frac{3}{2} u_{1,1} u_{1,3} u_{2,1} u_{2,3} u_{3,3} + \frac{3}{2} u_{1,2} u_{1,3} u_{2,2} u_{2,3} u_{3,3} + \frac{3}{4} u_{2,3}^2 u_{3,3} + \frac{3}{2} u_{1,3}^2 u_{2,3}^2 u_{3,3} +$$
$$\frac{3}{4} u_{2,1}^2 u_{2,3}^2 u_{3,3} + \frac{3}{2} u_{2,2} u_{2,3}^2 u_{3,3} + \frac{3}{4} u_{2,2}^2 u_{2,3}^2 u_{3,3} + \frac{3}{4} u_{2,3}^4 u_{3,3} + \frac{3}{2} u_{1,3} u_{3,1} u_{3,3} +$$
$$3 u_{1,1} u_{1,3} u_{3,1} u_{3,3} + \frac{9}{4} u_{1,1}^2 u_{1,3} u_{3,1} u_{3,3} + \frac{3}{4} u_{1,1}^3 u_{1,3} u_{3,1} u_{3,3} + \frac{3}{4} u_{1,2}^2 u_{1,3} u_{3,1} u_{3,3} +$$
$$\frac{3}{4} u_{1,1} u_{1,2}^2 u_{1,3} u_{3,1} u_{3,3} + \frac{3}{4} u_{1,3}^3 u_{3,1} u_{3,3} + \frac{3}{4} u_{1,1} u_{1,3}^3 u_{3,1} u_{3,3} + \frac{3}{4} u_{1,2} u_{1,3} u_{2,1} u_{3,1} u_{3,3} +$$
$$\frac{3}{4} u_{1,3} u_{2,1}^2 u_{3,1} u_{3,3} + \frac{3}{4} u_{1,1} u_{1,3} u_{2,1}^2 u_{3,1} u_{3,3} + \frac{3}{4} u_{1,2} u_{1,3} u_{2,1} u_{2,2} u_{3,1} u_{3,3} +$$
$$\frac{3}{4} u_{1,2} u_{2,3} u_{3,1} u_{3,3} + \frac{3}{4} u_{1,1} u_{1,2} u_{2,3} u_{3,1} u_{3,3} + \frac{9}{4} u_{2,1} u_{2,3} u_{3,1} u_{3,3} + \frac{3}{2} u_{1,1} u_{2,1} u_{2,3} u_{3,1} u_{3,3} +$$
$$\frac{3}{4} u_{1,1}^2 u_{2,1} u_{2,3} u_{3,1} u_{3,3} + \frac{3}{4} u_{1,3}^2 u_{2,1} u_{2,3} u_{3,1} u_{3,3} + \frac{3}{4} u_{2,1}^3 u_{2,3} u_{3,1} u_{3,3} +$$
$$\frac{3}{4} u_{1,2} u_{2,2} u_{2,3} u_{3,1} u_{3,3} + \frac{3}{4} u_{1,1} u_{1,2} u_{2,2} u_{2,3} u_{3,1} u_{3,3} + \frac{3}{2} u_{2,1} u_{2,2} u_{2,3} u_{3,1} u_{3,3} +$$
$$\frac{3}{4} u_{2,1} u_{2,2}^2 u_{2,3} u_{3,1} u_{3,3} + \frac{3}{4} u_{1,3} u_{2,3}^2 u_{3,1} u_{3,3} + \frac{3}{4} u_{1,1} u_{1,3} u_{2,3}^2 u_{3,1} u_{3,3} + \frac{3}{4} u_{2,1} u_{2,3}^3 u_{3,1} u_{3,3} +$$
$$\frac{3}{4} u_{3,1}^2 u_{3,3} + \frac{3}{2} u_{1,1} u_{3,1}^2 u_{3,3} + \frac{3}{4} u_{1,1}^2 u_{3,1}^2 u_{3,3} + \frac{3}{4} u_{1,3}^2 u_{3,1}^2 u_{3,3} + \frac{3}{4} u_{2,1}^2 u_{3,1}^2 u_{3,3} +$$
$$\frac{3}{4} u_{2,3}^2 u_{3,1}^2 u_{3,3} + \frac{3}{4} u_{1,3} u_{3,1}^3 u_{3,3} + \frac{3}{4} u_{1,1} u_{1,3} u_{3,1}^3 u_{3,3} + \frac{3}{4} u_{2,1} u_{2,3} u_{3,1}^3 u_{3,3} +$$
$$\frac{3}{4} u_{3,1}^4 u_{3,3} + \frac{9}{4} u_{1,2} u_{1,3} u_{3,2} u_{3,3} + \frac{3}{2} u_{1,1} u_{1,2} u_{1,3} u_{3,2} u_{3,3} + \frac{3}{4} u_{1,1}^2 u_{1,2} u_{1,3} u_{3,2} u_{3,3} +$$
$$\frac{3}{4} u_{1,2}^3 u_{1,3} u_{3,2} u_{3,3} + \frac{3}{4} u_{1,2} u_{1,3}^3 u_{3,2} u_{3,3} + \frac{3}{4} u_{1,3} u_{2,1} u_{3,2} u_{3,3} + \frac{3}{4} u_{1,1} u_{1,3} u_{2,1} u_{3,2} u_{3,3} +$$
$$\frac{3}{2} u_{1,2} u_{1,3} u_{2,2} u_{3,2} u_{3,3} + \frac{3}{4} u_{1,3} u_{2,1} u_{2,2} u_{3,2} u_{3,3} + \frac{3}{4} u_{1,1} u_{1,3} u_{2,1} u_{2,2} u_{3,2} u_{3,3} +$$

$$\frac{3}{4} u_{1,2} u_{1,3} u_{2,2}^2 u_{3,2} u_{3,3} + \frac{3}{2} u_{2,3} u_{3,2} u_{3,3} + \frac{3}{4} u_{1,2}^2 u_{2,3} u_{3,2} u_{3,3} + \frac{3}{4} u_{1,3}^2 u_{2,3} u_{3,2} u_{3,3} +$$
$$\frac{3}{4} u_{1,2} u_{2,1} u_{2,3} u_{3,2} u_{3,3} + \frac{3}{4} u_{1,1} u_{1,2} u_{2,1} u_{2,3} u_{3,2} u_{3,3} + \frac{3}{4} u_{2,1}^2 u_{2,3} u_{3,2} u_{3,3} +$$
$$3 u_{2,2} u_{2,3} u_{3,2} u_{3,3} + \frac{3}{4} u_{1,2}^2 u_{2,2} u_{2,3} u_{3,2} u_{3,3} + \frac{3}{4} u_{1,3}^2 u_{2,2} u_{2,3} u_{3,2} u_{3,3} +$$
$$\frac{3}{4} u_{2,1}^2 u_{2,2} u_{2,3} u_{3,2} u_{3,3} + \frac{9}{4} u_{2,2}^2 u_{2,3} u_{3,2} u_{3,3} + \frac{3}{4} u_{2,2}^3 u_{2,3} u_{3,2} u_{3,3} + \frac{3}{4} u_{1,2} u_{1,3} u_{2,3}^2 u_{3,2} u_{3,3} +$$
$$\frac{3}{4} u_{2,3}^3 u_{3,2} u_{3,3} + \frac{3}{4} u_{2,2} u_{2,3}^3 u_{3,2} u_{3,3} + \frac{3}{2} u_{1,2} u_{3,1} u_{3,2} u_{3,3} + \frac{3}{2} u_{1,1} u_{1,2} u_{3,1} u_{3,2} u_{3,3} +$$
$$\frac{3}{2} u_{2,1} u_{3,1} u_{3,2} u_{3,3} + \frac{3}{2} u_{2,1} u_{2,2} u_{3,1} u_{3,2} u_{3,3} + \frac{3}{4} u_{1,2} u_{1,3} u_{3,1}^2 u_{3,2} u_{3,3} +$$
$$\frac{3}{4} u_{2,3} u_{3,1}^2 u_{3,2} u_{3,3} + \frac{3}{4} u_{2,2} u_{2,3} u_{3,1}^2 u_{3,2} u_{3,3} + \frac{3}{4} u_{3,2}^2 u_{3,3} + \frac{3}{4} u_{1,2}^2 u_{3,2}^2 u_{3,3} +$$
$$\frac{3}{4} u_{1,3}^2 u_{3,2}^2 u_{3,3} + \frac{3}{2} u_{2,2} u_{3,2}^2 u_{3,3} + \frac{3}{4} u_{2,2}^2 u_{3,2}^2 u_{3,3} + \frac{3}{4} u_{2,3}^2 u_{3,2}^2 u_{3,3} + \frac{3}{4} u_{1,3} u_{3,1} u_{3,2}^2 u_{3,3} +$$
$$\frac{3}{4} u_{1,1} u_{1,3} u_{3,1} u_{3,2}^2 u_{3,3} + \frac{3}{4} u_{2,1} u_{2,3} u_{3,1} u_{3,2}^2 u_{3,3} + \frac{3}{2} u_{3,1}^2 u_{3,2}^2 u_{3,3} + \frac{3}{4} u_{1,2} u_{1,3} u_{3,2}^3 u_{3,3} +$$
$$\frac{3}{4} u_{2,3} u_{3,2}^3 u_{3,3} + \frac{3}{4} u_{2,2} u_{2,3} u_{3,2}^3 u_{3,3} + \frac{3}{4} u_{3,2}^4 u_{3,3} + \frac{15}{8} u_{1,3}^2 u_{3,3}^2 + \frac{3}{4} u_{1,1} u_{1,3}^2 u_{3,3}^2 +$$
$$\frac{3}{8} u_{1,1}^2 u_{1,3}^2 u_{3,3}^2 + \frac{3}{8} u_{1,2}^2 u_{1,3}^2 u_{3,3}^2 + \frac{3}{8} u_{1,3}^4 u_{3,3}^2 + \frac{3}{4} u_{1,2} u_{1,3} u_{2,3} u_{3,3}^2 + \frac{3}{4} u_{1,3} u_{2,1} u_{2,3} u_{3,3}^2 +$$
$$\frac{3}{4} u_{1,1} u_{1,3} u_{2,1} u_{2,3} u_{3,3}^2 + \frac{3}{4} u_{1,2} u_{1,3} u_{2,2} u_{2,3} u_{3,3}^2 + \frac{15}{8} u_{2,3}^2 u_{3,3}^2 + \frac{3}{4} u_{1,3}^2 u_{2,3}^2 u_{3,3}^2 +$$
$$\frac{3}{8} u_{2,1}^2 u_{2,3}^2 u_{3,3}^2 + \frac{3}{4} u_{2,2} u_{2,3}^2 u_{3,3}^2 + \frac{3}{8} u_{2,2}^2 u_{2,3}^2 u_{3,3}^2 + \frac{3}{8} u_{2,3}^4 u_{3,3}^2 + \frac{9}{4} u_{1,3} u_{3,1} u_{3,3}^2 +$$
$$\frac{9}{4} u_{1,1} u_{1,3} u_{3,1} u_{3,3}^2 + \frac{9}{4} u_{2,1} u_{2,3} u_{3,1} u_{3,3}^2 + \frac{15}{8} u_{3,1}^2 u_{3,3}^2 + \frac{3}{4} u_{1,1} u_{3,1}^2 u_{3,3}^2 + \frac{3}{8} u_{1,1}^2 u_{3,1}^2 u_{3,3}^2 +$$
$$\frac{3}{8} u_{1,3}^2 u_{3,1}^2 u_{3,3}^2 + \frac{3}{8} u_{2,1}^2 u_{3,1}^2 u_{3,3}^2 + \frac{3}{8} u_{2,3}^2 u_{3,1}^2 u_{3,3}^2 + \frac{3}{8} u_{3,1}^4 u_{3,3}^2 + \frac{9}{4} u_{1,2} u_{1,3} u_{3,2} u_{3,3}^2 +$$
$$\frac{9}{4} u_{2,3} u_{3,2} u_{3,3}^2 + \frac{9}{4} u_{2,2} u_{2,3} u_{3,2} u_{3,3}^2 + \frac{3}{4} u_{1,2} u_{3,1} u_{3,2} u_{3,3}^2 + \frac{3}{4} u_{1,1} u_{1,2} u_{3,1} u_{3,2} u_{3,3}^2 +$$
$$\frac{3}{4} u_{2,1} u_{3,1} u_{3,2} u_{3,3}^2 + \frac{3}{4} u_{2,1} u_{2,2} u_{3,1} u_{3,2} u_{3,3}^2 + \frac{15}{8} u_{3,2}^2 u_{3,3}^2 + \frac{3}{8} u_{1,2}^2 u_{3,2}^2 u_{3,3}^2 +$$
$$\frac{3}{8} u_{1,3}^2 u_{3,2}^2 u_{3,3}^2 + \frac{3}{4} u_{2,2} u_{3,2}^2 u_{3,3}^2 + \frac{3}{8} u_{2,2}^2 u_{3,2}^2 u_{3,3}^2 + \frac{3}{8} u_{2,3}^2 u_{3,2}^2 u_{3,3}^2 + \frac{3}{4} u_{3,1}^2 u_{3,2}^2 u_{3,3}^2 +$$
$$\frac{3}{8} u_{3,2}^4 u_{3,3}^2 + u_{3,3}^3 + \frac{3}{2} u_{1,3}^2 u_{3,3}^3 + \frac{3}{2} u_{2,3}^2 u_{3,3}^3 + \frac{3}{4} u_{1,3} u_{3,1} u_{3,3}^3 + \frac{3}{4} u_{1,1} u_{1,3} u_{3,1} u_{3,3}^3 +$$
$$\frac{3}{4} u_{2,1} u_{2,3} u_{3,1} u_{3,3}^3 + \frac{3}{2} u_{3,1}^2 u_{3,3}^3 + \frac{3}{4} u_{1,2} u_{1,3} u_{3,2} u_{3,3}^3 + \frac{3}{4} u_{2,3} u_{3,2} u_{3,3}^3 + \frac{3}{4} u_{2,2} u_{2,3} u_{3,2} u_{3,3}^3 +$$

$$\frac{3}{2} u_{3,2}^2 u_{3,3}^3 + \frac{3 u_{3,3}^4}{2} + \frac{3}{8} u_{1,3}^2 u_{3,3}^4 + \frac{3}{8} u_{2,3}^2 u_{3,3}^4 + \frac{3}{8} u_{3,1}^2 u_{3,3}^4 + \frac{3}{8} u_{3,2}^2 u_{3,3}^4 + \frac{3 u_{3,3}^5}{4} + \frac{u_{3,3}^6}{8}$$

In[28]:=

$$\mathcal{E}_{SS} = \mu\, I_2 + \frac{1}{3} \mathcal{A}\, I_3 + \mathcal{D}\, I_2{}^2;$$

In[29]:=

$$\mathcal{U}_{SS} = \text{Expand}[\mathcal{E}_{SS}]$$

Out[29]=

$$\mu u_{1,1}^2 + \frac{1}{3} \mathcal{A} u_{1,1}^3 + \mu u_{1,1}^3 + \frac{1}{2} \mathcal{A} u_{1,1}^4 + \mathcal{D} u_{1,1}^4 + \frac{1}{4} \mu u_{1,1}^4 + \frac{1}{4} \mathcal{A} u_{1,1}^5 + 2 \mathcal{D} u_{1,1}^5 + \frac{1}{24} \mathcal{A} u_{1,1}^6 + \cdots 2742 \cdots + \frac{3}{2} \mathcal{D} u_{3,2}^2 u_{3,3}^5 + \frac{1}{24} \mathcal{A} u_{3,3}^6 + \frac{3}{2} \mathcal{D} u_{3,3}^6 + \frac{1}{4} \mathcal{D} u_{1,3}^2 u_{3,3}^6 + \frac{1}{4} \mathcal{D} u_{2,3}^2 u_{3,3}^6 + \frac{1}{4} \mathcal{D} u_{3,1}^2 u_{3,3}^6 + \frac{1}{4} \mathcal{D} u_{3,2}^2 u_{3,3}^6 + \frac{1}{2} \mathcal{D} u_{3,3}^7 + \frac{1}{16} \mathcal{D} u_{3,3}^8$$

large output | show less | show more | show all | set size limit...

Cauchy stress

The Cauchy stress is the gradient of the strain energy w.r.t. deformation gradient

In[30]:=

```
σ[𝒰_] := Monitor[Table[Simplify[D[𝒰, u_{i,j}]], {i, d}, {j, d}], {i, j}]
```

Linear elastic

In[31]:=

```
σ_LE = σ[𝒰_LE]; TableForm[σ_LE]
```

Out[31]//TableForm=

$(\lambda + 2\mu) u_{1,1} + \lambda (u_{2,2} + u_{3,3})$	$\mu (u_{1,2} + u_{2,1})$	$\mu (u_{1,3} + u_{3,1})$
$\mu (u_{1,2} + u_{2,1})$	$\lambda u_{1,1} + (\lambda + 2\mu) u_{2,2} + \lambda u_{3,3}$	$\mu (u_{2,3} + u_{3,2})$
$\mu (u_{1,3} + u_{3,1})$	$\mu (u_{2,3} + u_{3,2})$	$\lambda u_{1,1} + \lambda u_{2,2} + (\lambda + 2\mu) u_{3,3}$

In terms of the deformation gradient components

In[32]:=

```
σ𝔽_LE = σ_LE /. Flatten[Table[u_{i,j} → F_{idx2[i,j]} - δ[[i, j]], {i, d}, {j, d}]];
TableForm[σ𝔽_LE]
```

Out[32]//TableForm=

$(\lambda + 2\mu)(-1 + F_{11}) + \lambda(-2 + F_{22} + F_{33})$	$\mu (F_{12} + F_{21})$	$\mu (F_{13} + F_{31})$
$\mu (F_{12} + F_{21})$	$\lambda(-1 + F_{11}) + (\lambda + 2\mu)(-1 + F_{22}) + \lambda(-1 + F_{33})$	$\mu (F_{23} + F_{32})$
$\mu (F_{13} + F_{31})$	$\mu (F_{23} + F_{32})$	$\lambda(-1 + F_{11}) +$,

In[33]:=
```
sigmaFLE = σ𝔽LE;
DumpSave["sigmaF_LE.mx", sigmaFLE];
```

Isotropic soft solid shear modes

In[35]:=
```
σSS = σ[𝒰SS];
```

The expressions for the stress components are very long. Individual components can be accessed as follows:

In[36]:=
```
σSS[[1, 1]] /. {𝒜 → 0, 𝒟 → 0}
```

Out[36]=

$$\frac{1}{4}\left(12\,\mu\,u_{1,1}^2 + 4\,\mu\,u_{1,1}^3 + 4\,\mu\,u_{1,2}^2 + 4\,\mu\,u_{1,3}^2 + 4\,\mu\,u_{2,1}^2 + 4\,\mu\,u_{1,3}\,u_{2,1}\,u_{2,3} + 4\,\mu\,u_{1,3}\,u_{3,1} + 4\,\mu\,u_{3,1}^2 + u_{1,1}\left(8\,\mu + 4\,\mu\,u_{1,2}^2 + 4\,\mu\,u_{1,3}^2 + 4\,\mu\,u_{2,1}^2 + 4\,\mu\,u_{3,1}^2\right) + u_{1,2}\left(u_{2,1}\left(4\,\mu + 4\,\mu\,u_{2,2}\right) + 4\,\mu\,u_{3,1}\,u_{3,2}\right) + 4\,\mu\,u_{1,3}\,u_{3,1}\,u_{3,3}\right)$$

In[37]:=
```
σ𝔽SS = σSS /. Flatten[Table[u_{i,j} → F_{idx2[i,j]} - δ[[i, j]], {i, d}, {j, d}]];
```

In[38]:=
```
sigmaFSS = σ𝔽SS;
DumpSave["sigmaF_SS.mx", sigmaFSS];
```

Linearly polarized shear wave

To check above computations, extract expressions for a linearly polarized shear wave in which only $u_{1,3}$ is non-zero

In[40]:=
```
setZero = Flatten[Table[u_{i,j} → 0, {i, d}, {j, d}]]
```

Out[40]=
```
{u1,1 → 0, u1,2 → 0, u1,3 → 0, u2,1 → 0, u2,2 → 0, u2,3 → 0, u3,1 → 0, u3,2 → 0, u3,3 → 0}
```

In[41]:=
```
setZeroNot13 = Drop[setZero, {3, 3}]
```

Out[41]=
```
{u1,1 → 0, u1,2 → 0, u2,1 → 0, u2,2 → 0, u2,3 → 0, u3,1 → 0, u3,2 → 0, u3,3 → 0}
```

In[42]:=
```
Simplify[σSS /. setZero]
```

Out[42]=
```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

In[43]:=
```
σSSLP = Simplify[σSS /. setZeroNot13]; TableForm[σSSLP[[1, 3]]]
```

Out[43]//TableForm=

$$\frac{1}{4}\left(4\,\mu\,u_{1,3}+2\,(\mathcal{A}+2\,(\mathcal{D}+\mu))\,u_{1,3}^{3}+(\mathcal{A}+6\,\mathcal{D})\,u_{1,3}^{5}+2\,\mathcal{D}\,u_{1,3}^{7}\right)$$

Momentum conservation is stated as $\rho\,v_{1,t}=\sigma_{1\,j,j}=\partial_3\left[\,\mu\,u_{1,3}+\left(\mu+\frac{\mathcal{A}}{2}+\mathcal{D}\right)u_{1,3}^{3}+\frac{1}{4}\,(\mathcal{A}+6\,\mathcal{D})\,u_{1,3}^{5}+\frac{1}{2}\,\mathcal{D}\,u_{1,3}^{7}\right]$, which differs from Zabolotskaya et al (2004) in the terms $u_{1,3}^{5}$ and $u_{1,3}^{7}$. Verify result against different computational steps.

In[44]:=
```
setZeroD23 = Flatten[Table[u_{i,j} → 0, {i, d}, {j, 1, 2}]]
```

Out[44]=

$$\{u_{1,1}\to 0,\ u_{1,2}\to 0,\ u_{2,1}\to 0,\ u_{2,2}\to 0,\ u_{3,1}\to 0,\ u_{3,2}\to 0\}$$

In[45]:=
```
setZeroU23 = Flatten[Table[u_{i,j} → 0, {i, 2, d}, {j, 1, d}]]
```

Out[45]=

$$\{u_{2,1}\to 0,\ u_{2,2}\to 0,\ u_{2,3}\to 0,\ u_{3,1}\to 0,\ u_{3,2}\to 0,\ u_{3,3}\to 0\}$$

In[46]:=
```
𝒰SSLP = 𝒰SS /. Flatten[{setZeroD23, setZeroU23}]
```

Out[46]=

$$\frac{1}{2}\,\mu\,u_{1,3}^{2}+\frac{1}{8}\,\mathcal{A}\,u_{1,3}^{4}+\frac{1}{4}\,\mathcal{D}\,u_{1,3}^{4}+\frac{1}{4}\,\mu\,u_{1,3}^{4}+\frac{1}{24}\,\mathcal{A}\,u_{1,3}^{6}+\frac{1}{4}\,\mathcal{D}\,u_{1,3}^{6}+\frac{1}{16}\,\mathcal{D}\,u_{1,3}^{8}$$

In[47]:=
```
σSSLP2 = σ[𝒰SSLP]; TableForm[σSSLP2]
```

Out[47]//TableForm=

0	0	$\frac{1}{4}\left(4\,\mu\,u_{1,3}+2\,(\mathcal{A}+2\,(\mathcal{D}+\mu))\,u_{1,3}^{3}+(\mathcal{A}+6\,\mathcal{D})\,u_{1,3}^{5}+2\,\mathcal{D}\,u_{1,3}^{7}\right)$
0	0	0
0	0	0

In[48]:=
```
Simplify[D[𝒰SSLP, u_{1,3}]]
```

Out[48]=

$$\frac{1}{4}\left(4\,\mu\,u_{1,3}+2\,(\mathcal{A}+2\,(\mathcal{D}+\mu))\,u_{1,3}^{3}+(\mathcal{A}+6\,\mathcal{D})\,u_{1,3}^{5}+2\,\mathcal{D}\,u_{1,3}^{7}\right)$$

Elliptically polarized shear wave

As a further check, extract expressions for an elliptically polarized shear wave in which only $u_{1,3}$, $u_{2,3}$ are non-zero

In[49]:=
```
setZero = Flatten[Table[u_{i,j} → 0, {i, d}, {j, d}]]
```

Out[49]=

$$\{u_{1,1}\to 0,\ u_{1,2}\to 0,\ u_{1,3}\to 0,\ u_{2,1}\to 0,\ u_{2,2}\to 0,\ u_{2,3}\to 0,\ u_{3,1}\to 0,\ u_{3,2}\to 0,\ u_{3,3}\to 0\}$$

In[50]:=
```
setZeroNot1323 = Drop[Drop[setZero, {3, 3}], {5, 5}]
```

Out[50]= $\{u_{1,1} \to 0,\ u_{1,2} \to 0,\ u_{2,1} \to 0,\ u_{2,2} \to 0,\ u_{3,1} \to 0,\ u_{3,2} \to 0,\ u_{3,3} \to 0\}$

In[51]:=
```
σSSEP = Simplify[σSS /. setZeroNot1323]; TableForm[σSSEP[[1 ;; 2, 3]]]
```

Out[51]//TableForm=

$$\frac{1}{4} u_{1,3} \left(4 \mu + 2 \mathcal{D} u_{1,3}^6 + 2 (\mathcal{A} + 2 (\mathcal{D} + \mu)) u_{2,3}^2 + (\mathcal{A} + 6 \mathcal{D}) u_{2,3}^4 + 2 \mathcal{D} u_{2,3}^6 + u_{1,3}^4 \left(\mathcal{A} + 6 \mathcal{D} + 6 \mathcal{D} u_{2,3}^2\right) + 2\right.$$

$$\frac{1}{4} u_{2,3} \left(4 \mu + 2 \mathcal{D} u_{1,3}^6 + 2 (\mathcal{A} + 2 (\mathcal{D} + \mu)) u_{2,3}^2 + (\mathcal{A} + 6 \mathcal{D}) u_{2,3}^4 + 2 \mathcal{D} u_{2,3}^6 + u_{1,3}^4 \left(\mathcal{A} + 6 \mathcal{D} + 6 \mathcal{D} u_{2,3}^2\right) + 2\right.$$

The above again differs from Zabolotskaya et al (2004). Alternative computational steps

In[52]:=
```
setZeroD23 = Flatten[Table[u_{i,j} → 0, {i, d}, {j, 1, 2}]]
```

Out[52]= $\{u_{1,1} \to 0,\ u_{1,2} \to 0,\ u_{2,1} \to 0,\ u_{2,2} \to 0,\ u_{3,1} \to 0,\ u_{3,2} \to 0\}$

In[53]:=
```
setZeroU3 = Flatten[Table[u_{i,j} → 0, {i, 3, d}, {j, 1, d}]]
```

Out[53]= $\{u_{3,1} \to 0,\ u_{3,2} \to 0,\ u_{3,3} \to 0\}$

In[54]:=
```
𝒰SSEP = 𝒰SS /. Flatten[{setZeroD23, setZeroU3}]
```

Out[54]=

$$\frac{1}{2} \mu u_{1,3}^2 + \frac{1}{8} \mathcal{A} u_{1,3}^4 + \frac{1}{4} \mathcal{D} u_{1,3}^4 + \frac{1}{4} \mu u_{1,3}^4 + \frac{1}{24} \mathcal{A} u_{1,3}^6 + \frac{1}{4} \mathcal{D} u_{1,3}^6 + \frac{1}{16} \mathcal{D} u_{1,3}^8 +$$
$$\frac{1}{2} \mu u_{2,3}^2 + \frac{1}{4} \mathcal{A} u_{1,3}^2 u_{2,3}^2 + \frac{1}{2} \mathcal{D} u_{1,3}^2 u_{2,3}^2 + \frac{1}{2} \mu u_{1,3}^2 u_{2,3}^2 + \frac{1}{8} \mathcal{A} u_{1,3}^4 u_{2,3}^2 +$$
$$\frac{3}{4} \mathcal{D} u_{1,3}^4 u_{2,3}^2 + \frac{1}{4} \mathcal{D} u_{1,3}^6 u_{2,3}^2 + \frac{1}{8} \mathcal{A} u_{2,3}^4 + \frac{1}{4} \mathcal{D} u_{2,3}^4 + \frac{1}{4} \mu u_{2,3}^4 + \frac{1}{8} \mathcal{A} u_{1,3}^2 u_{2,3}^4 +$$
$$\frac{3}{4} \mathcal{D} u_{1,3}^2 u_{2,3}^4 + \frac{3}{8} \mathcal{D} u_{1,3}^4 u_{2,3}^4 + \frac{1}{24} \mathcal{A} u_{2,3}^6 + \frac{1}{4} \mathcal{D} u_{2,3}^6 + \frac{1}{4} \mathcal{D} u_{1,3}^2 u_{2,3}^6 + \frac{1}{16} \mathcal{D} u_{2,3}^8$$

In[55]:=
```
σSSEP2 = σ[𝒰SSEP]; TableForm[σSSEP2]
```

Out[55]//TableForm=

0	0	$\frac{1}{4} u_{1,3} \left(4 \mu + 2 \mathcal{D} u_{1,3}^6 + 2 (\mathcal{A} + 2 (\mathcal{D} + \mu)) u_{2,3}^2 + (\mathcal{A} + 6 \mathcal{D}) u_{2,3}^4 + 2 \mathcal{D} u_{2,3}^6 + u_{1,3}^4 (\mathcal{A} + 6 \mathcal{D} + 6\right.$
0	0	$\frac{1}{4} u_{2,3} \left(4 \mu + 2 \mathcal{D} u_{1,3}^6 + 2 (\mathcal{A} + 2 (\mathcal{D} + \mu)) u_{2,3}^2 + (\mathcal{A} + 6 \mathcal{D}) u_{2,3}^4 + 2 \mathcal{D} u_{2,3}^6 + u_{1,3}^4 (\mathcal{A} + 6 \mathcal{D} + 6\right.$
0	0	0

In[56]:=
```
Simplify[D[𝒰SSEP, u1,3]]
```

Out[56]=
$$\frac{1}{4} u_{1,3} \left(4 \mu + 2 \mathcal{D} u_{1,3}^6 + 2 (\mathcal{A} + 2 (\mathcal{D} + \mu)) u_{2,3}^2 + (\mathcal{A} + 6 \mathcal{D}) u_{2,3}^4 + 2 \mathcal{D} u_{2,3}^6 + u_{1,3}^4 (\mathcal{A} + 6 \mathcal{D} + 6 \mathcal{D} u_{2,3}^2) + 2 u_{1,3}^2 (\mathcal{A} + 2 (\mathcal{D} + \mu) + (\mathcal{A} + 6 \mathcal{D}) u_{2,3}^2 + 3 \mathcal{D} u_{2,3}^4)\right)$$

In[57]:=
```
Simplify[D[𝒰SSEP, u2,3]]
```

Out[57]=
$$\frac{1}{4} u_{2,3} \left(4 \mu + 2 \mathcal{D} u_{1,3}^6 + 2 (\mathcal{A} + 2 (\mathcal{D} + \mu)) u_{2,3}^2 + (\mathcal{A} + 6 \mathcal{D}) u_{2,3}^4 + 2 \mathcal{D} u_{2,3}^6 + u_{1,3}^4 (\mathcal{A} + 6 \mathcal{D} + 6 \mathcal{D} u_{2,3}^2) + 2 u_{1,3}^2 (\mathcal{A} + 2 (\mathcal{D} + \mu) + (\mathcal{A} + 6 \mathcal{D}) u_{2,3}^2 + 3 \mathcal{D} u_{2,3}^4)\right)$$

Tensions

Tensions (forces per unit volume) in the directions of the reference coordinates are obtained by taking divergence of Cauchy stress

Utility functions

Define differentiation of deformation tensor

In[58]:=
```
dF[σF_, j_] :=
 Simplify[D[σF /. Flatten[Table[ Fidx2[k,l] -> Fidx2[k,l][x], {k, d}, {l, d}]], x] /.
   Flatten[{Table[ Fidx2[k,l]'[x] -> Fidx2[k,l],j, {k, d}, {l, d}],
     Table[ Fidx2[k,l][x] -> Fidx2[k,l], {k, d}, {l, d}]}]]
```

Define divergence of stress tensor expressed in terms of 𝔽

In[59]:=
```
𝕗[σ_] := Monitor[Table[Sum[dF[σ[[i, j]], j], {j, d}], {i, d}], {i, j}]
```

Replace 𝔽 and spatial derivative of 𝔽 with 𝕦

In[60]:=
```
𝕦𝔽[σ_] := σ /. Flatten[{
    Table[ Fidx2[k,l] → uk,l, {k, d}, {l, d}],
    Table[ Fidx2[k,l],m → uk,idx2[Min[{l,m}],Max[{l,m}]], {k, d}, {l, d}, {m, d}]
  }]
```

Tensions for linear elastic solid

In[61]:=
```
𝕗LE = 𝕗[σ𝔽LE]; TableForm[𝕗LE]
```

Out[61]//TableForm=

$$(\lambda + 2\,\mu)\,F_{11,1} + \mu\,(F_{12,2} + F_{21,2}) + \mu\,(F_{13,3} + F_{31,3}) + \lambda\,(F_{22,1} + F_{33,1})$$
$$\lambda\,F_{11,2} + \mu\,(F_{12,1} + F_{21,1}) + (\lambda + 2\,\mu)\,F_{22,2} + \mu\,(F_{23,3} + F_{32,3}) + \lambda\,F_{33,2}$$
$$\lambda\,F_{11,3} + \lambda\,F_{22,3} + \mu\,(F_{13,1} + F_{31,1}) + \mu\,(F_{23,2} + F_{32,2}) + (\lambda + 2\,\mu)\,F_{33,3}$$

In[62]:=
```
fFLE = 𝕗LE;
DumpSave["fF_LE.mx", fFLE];
```

Verify against vector formula $(\lambda+\mu)\nabla\,\nabla\cdot u + \mu\,\nabla^2 u$

In[64]:=
```
𝕗𝕦LE = Map[𝕦𝔽[#] &, 𝕗LE]; TableForm[𝕗𝕦LE]
```

Out[64]//TableForm=

$$(\lambda + 2\,\mu)\,u_{1,11} + \mu\,(u_{1,22} + u_{2,12}) + \mu\,(u_{1,33} + u_{3,13}) + \lambda\,(u_{2,12} + u_{3,13})$$
$$\lambda\,u_{1,12} + \mu\,(u_{1,12} + u_{2,11}) + (\lambda + 2\,\mu)\,u_{2,22} + \lambda\,u_{3,23} + \mu\,(u_{2,33} + u_{3,23})$$
$$\lambda\,u_{1,13} + \lambda\,u_{2,23} + \mu\,(u_{1,13} + u_{3,11}) + \mu\,(u_{2,23} + u_{3,22}) + (\lambda + 2\,\mu)\,u_{3,33}$$

In[65]:=
```
form = Table[Simplify[Sum[(λ + μ) u_{j,idx2[Min[{i,j}],Max[{i,j}]]} + μ u_{i,idx2[j,j]}, {j, d}]], {i, d}]
```

Out[65]=

$$\{\mu\,u_{1,11} + (\lambda + \mu)\,u_{1,11} + \mu\,u_{1,22} + \mu\,u_{1,33} + (\lambda + \mu)\,u_{2,12} + (\lambda + \mu)\,u_{3,13},$$
$$(\lambda + \mu)\,u_{1,12} + \mu\,u_{2,11} + \mu\,u_{2,22} + (\lambda + \mu)\,u_{2,22} + \mu\,u_{2,33} + (\lambda + \mu)\,u_{3,23},$$
$$(\lambda + \mu)\,u_{1,13} + (\lambda + \mu)\,u_{2,23} + \mu\,u_{3,11} + \mu\,u_{3,22} + \lambda\,u_{3,33} + 2\,\mu\,u_{3,33}\}$$

In[66]:=
```
Simplify[𝕗𝕦LE - form]
```

Out[66]=
```
{0, 0, 0}
```

Tensions for isotropic soft solid shear modes

In[67]:=
```
𝕗SS = 𝕗[σ𝔽SS];
```

In[68]:=
```
fFSS = 𝕗SS;
DumpSave["fF_SS.mx", fFSS];
```

Flux Jacobians

For an incompressible medium in which all model parameters are scaled by the density and deformations are sufficiently small to enable use of the Cauchy stress tensor (as opposed to the first Piola-

Kirchhoff stress), the equations of motion $v_{i,t} = \sigma_{i\,j,j}$ coupled with rate of deformation equations $F_{i\,j,j} = v_{i,j}$ form a conservative system $q_t - \nabla \cdot f = 0$ with $q = (v\;\; F)^T$, $f = (\sigma\;\; v \otimes \delta)^T$, or in component form $v_{i,t} = \sigma_{i\,k,k}$, $F_{i\,j,t} = \left(v_i\, \delta_{j\,k}\right)_{,k}$.

To use a wave propagation approach to the numerical solution of the equations of motion, the eigenmodes of the system flux Jacobians are required

In[ ]:=
```
f[σ_, v_] := Transpose[Join[σ, Flatten[Outer[Times, v, δ], 1]]];
```

In[ ]:=
```
sig = Table[σ_idx2[i,j], {i, d}, {j, d}];
vel = Table[v_i, {i, d}];
flux = f[sig, vel]; TableForm[Transpose[flux]]
```

Out[ ]//TableForm=

σ_{11}	σ_{12}	σ_{13}
σ_{21}	σ_{22}	σ_{23}
σ_{31}	σ_{32}	σ_{33}
v_1	0	0
0	v_1	0
0	0	v_1
v_2	0	0
0	v_2	0
0	0	v_2
v_3	0	0
0	v_3	0
0	0	v_3

Define gradient w.r.t. q

In[ ]:=
```
q = Flatten[{Table[v_i, {i, d}], Table[F_idx2[i,j], {i, d}, {j, d}]}];
nq = Length[q]; TableForm[q, TableDirections → Row]
```

Out[ ]//TableForm=

v_1 v_2 v_3 F_{11} F_{12} F_{13} F_{21} F_{22} F_{23} F_{31} F_{32} F_{33}

In[ ]:=
```
gradq[f_] := Monitor[Table[D[f, q[[α]]], {α, nq}], α]
```

Linear elastic

In[ ]:=
```
f_LE = f[σ𝔽_LE, vel]; TableForm[f_LE, TableDirections → Row]
```

Out[ ]//TableForm=

$(\lambda+2\mu)(-1+F_{11})+\lambda(-2+F_{22}+F_{33})$	$\mu(F_{12}+F_{21})$	$\mu(F_{13}+F_{31})$
$\mu(F_{12}+F_{21})$	$\lambda(-1+F_{11})+(\lambda+2\mu)(-1+F_{22})+\lambda(-1+F_{33})$	$\mu(F_{23}+F_{32})$
$\mu(F_{13}+F_{31})$	$\mu(F_{23}+F_{32})$	$\lambda(-1+F_{11})+\lambda(-1+$
v_1	0	0
0	v_1	0
0	0	v_1
v_2	0	0
0	v_2	0
0	0	v_2
v_3	0	0
0	v_3	0
0	0	v_3

Flux Jacobians

In[ ]:=
```
A_LE = Map[Transpose[gradq[#]] &, f_LE]; TableForm[A_LE[[1]]]
```

Out[ ]//TableForm=

0	0	0	$\lambda+2\mu$	0	0	0	λ	0	0	0	λ
0	0	0	0	μ	0	μ	0	0	0	0	0
0	0	0	0	0	μ	0	0	0	μ	0	0
1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

In[ ]:=
```
TableForm[A_LE[[2]]]
```

Out[ ]//TableForm=

0	0	0	0	μ	0	μ	0	0	0	0	0
0	0	0	λ	0	0	0	$\lambda + 2\mu$	0	0	0	λ
0	0	0	0	0	0	0	0	μ	0	μ	0
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

In[ ]:=
```
TableForm[A_LE[[3]]]
```

Out[ ]//TableForm=

0	0	0	0	0	μ	0	0	0	μ	0	0
0	0	0	0	0	0	0	0	μ	0	μ	0
0	0	0	λ	0	0	0	λ	0	0	0	$\lambda + 2\mu$
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0

In[ ]:=
```
fLE = f_LE; ALE = A_LE;
DumpSave["qfA_LE.mx", {q, fLE, ALE}];
```

Eigensystem - eigenvalues

In[ ]:=
```
ΛR_LE = Map[Eigensystem[#] &, A_LE] /. {Sqrt[λ + 2 μ] → c_P, Sqrt[μ] → c_S};
Λ_LE = Table[ΛR_LE[[i, 1]], {i, d}];
R_LE = Table[ΛR_LE[[i, 2]], {i, d}];
TableForm[Λ_LE]
```

Out[ ]//TableForm=

0	0	0	0	0	0	$-c_S$	$-c_S$	c_S	c_S	$-c_P$	c_P
0	0	0	0	0	0	$-c_S$	$-c_S$	c_S	c_S	$-c_P$	c_P
0	0	0	0	0	0	$-c_S$	$-c_S$	c_S	c_S	$-c_P$	c_P

Eigensystem - eigenvectors (displayed as rows in Mathematica)

In[ ]:=
```
TableForm[R_LE[[1]]]
```

Out[ ]//TableForm=

0	0	0	$-\frac{\lambda}{\lambda+2\,\mu}$	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	-1	0	0	0	1	0	0
0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	$-\frac{\lambda}{\lambda+2\,\mu}$	0	0	0	1	0	0	0	0
0	0	0	0	-1	0	1	0	0	0	0	0
0	0	$-c_S$	0	0	0	0	0	0	1	0	0
0	$-c_S$	0	0	0	0	1	0	0	0	0	0
0	0	c_S	0	0	0	0	0	0	1	0	0
0	c_S	0	0	0	0	1	0	0	0	0	0
$-c_P$	0	0	1	0	0	0	0	0	0	0	0
c_P	0	0	1	0	0	0	0	0	0	0	0

In[ ]:=
```
TableForm[R_LE[[2]]]
```

Out[ ]//TableForm=

0	0	0	-1	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	-1	0	1	0
0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	$-\frac{\lambda+2\,\mu}{\lambda}$	0	0	0	1	0	0	0	0
0	0	0	0	-1	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0
0	0	$-c_S$	0	0	0	0	0	0	0	1	0
$-c_S$	0	0	0	1	0	0	0	0	0	0	0
0	0	c_S	0	0	0	0	0	0	0	1	0
c_S	0	0	0	1	0	0	0	0	0	0	0
0	$-c_P$	0	0	0	0	0	1	0	0	0	0
0	c_P	0	0	0	0	0	1	0	0	0	0

In[ ]:=
```
TableForm[R_LE[[3]]]
```
Out[ ]//TableForm=

0	0	0	$-\frac{\lambda+2\mu}{\lambda}$	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	-1	0	1	0
0	0	0	0	0	-1	0	0	0	1	0	0
0	0	0	-1	0	0	0	1	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0
0	$-c_S$	0	0	0	0	0	0	1	0	0	0
$-c_S$	0	0	0	0	1	0	0	0	0	0	0
0	c_S	0	0	0	0	0	0	1	0	0	0
c_S	0	0	0	0	1	0	0	0	0	0	0
0	0	$-c_P$	0	0	0	0	0	0	0	0	1
0	0	c_P	0	0	0	0	0	0	0	0	1

Verify eigensystem

In[ ]:=
```
TableForm[Table[A_LE[[i]].R_LE[[i, α]] - Λ_LE[[i, α]] R_LE[[i, α]] /. {c_P → √(λ + 2 μ), c_S → √μ},
  {α, nq}, {i, d}], TableDirections → Row]
```
Out[ ]//TableForm=

0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0

Decomposition of δq onto eigenbasis, R c = δq

In[ ]:=
```
δ𝕢 = Table[δq_α, {α, nq}]; TableForm[δ𝕢, TableDirections → Row]
```
Out[ ]//TableForm=

δq_1 δq_2 δq_3 δq_4 δq_5 δq_6 δq_7 δq_8 δq_9 δq_{10} δq_{11} δq_{12}

Define coefficients c

In[ ]:=
```
𝕔 = Table[LinearSolve[Transpose[R[[i]]], δ𝕢], {i, d}];
```

Select propagating waves (place arbitrary positive values for wave velocities into eigenvalues)

In[ ]:=
```
Λ12 = Λ /. {c_S → 1, c_P → 2};
iW = Table[Select[Table[α, {α, nq}], Λ12[[i, #]] ≠ 0 &], {i, d}]; TableForm[iW]
```
Out[ ]//TableForm=

7	8	9	10	11	12
7	8	9	10	11	12
7	8	9	10	11	12

Number of propagating waves. Should be the same for all d spatial dimensions.

In[ ]:=
```
nW = Length[iW[[1]]]
```
Out[ ]=
```
6
```

In[ ]:=
```
λW = Table[Λ[[i, iW[[i]]]], {i, d}]; TableForm[λW]
```
Out[ ]//TableForm=

$-c_S$	$-c_S$	c_S	c_S	$-c_P$	c_P
$-c_S$	$-c_S$	c_S	c_S	$-c_P$	c_P
$-c_S$	$-c_S$	c_S	c_S	$-c_P$	c_P

In[ ]:=
```
W = Table[R[[i, iW[[i]]]], {i, d}]; TableForm[W[[1]]]
```
Out[ ]//TableForm=

0	0	$-c_S$	0	0	0	0	0	0	1	0	0
0	$-c_S$	0	0	0	0	1	0	0	0	0	0
0	0	c_S	0	0	0	0	0	0	1	0	0
0	c_S	0	0	0	0	1	0	0	0	0	0
$-c_P$	0	0	1	0	0	0	0	0	0	0	0
c_P	0	0	1	0	0	0	0	0	0	0	0

In[ ]:=
```
TableForm[W[[2]]]
```
Out[ ]//TableForm=

0	0	$-c_S$	0	0	0	0	0	0	0	1	0
$-c_S$	0	0	0	1	0	0	0	0	0	0	0
0	0	c_S	0	0	0	0	0	0	0	1	0
c_S	0	0	0	1	0	0	0	0	0	0	0
0	$-c_P$	0	0	0	0	0	1	0	0	0	0
0	c_P	0	0	0	0	0	1	0	0	0	0

In[ ]:=
```
TableForm[W[[3]]]
```
Out[ ]//TableForm=

0	$-c_S$	0	0	0	0	0	0	1	0	0	0
$-c_S$	0	0	0	0	1	0	0	0	0	0	0
0	c_S	0	0	0	0	0	0	1	0	0	0
c_S	0	0	0	0	1	0	0	0	0	0	0
0	0	$-c_P$	0	0	0	0	0	0	0	0	1
0	0	c_P	0	0	0	0	0	0	0	0	1

In[ ]:=
```
cW = Table[ℭ[[i, iW[[i]]]], {i, d}]; TableForm[cW]
```

Out[ ]//TableForm=

$\frac{-\delta q_3+c_S\,\delta q_6+c_S\,\delta q_{10}}{2\,c_S}$	$\frac{-\delta q_2+c_S\,\delta q_5+c_S\,\delta q_7}{2\,c_S}$	$\frac{\delta q_3+c_S\,\delta q_6+c_S\,\delta q_{10}}{2\,c_S}$	$\frac{\delta q_2+c_S\,\delta q_5+c_S\,\delta q_7}{2\,c_S}$	$\frac{-\lambda\,\delta q_1-2\,\mu\,\delta q_1+\lambda\,c_P\,\delta q_4+2}{2\,(\lambda+}$
$\frac{-\delta q_3+c_S\,\delta q_9+c_S\,\delta q_{11}}{2\,c_S}$	$\frac{-\delta q_1+c_S\,\delta q_5+c_S\,\delta q_7}{2\,c_S}$	$\frac{\delta q_3+c_S\,\delta q_9+c_S\,\delta q_{11}}{2\,c_S}$	$\frac{\delta q_1+c_S\,\delta q_5+c_S\,\delta q_7}{2\,c_S}$	$\frac{-\lambda\,\delta q_2-2\,\mu\,\delta q_2+\lambda\,c_P\,\delta q_4+\lambda}{2\,(\lambda+}$
$\frac{-\delta q_2+c_S\,\delta q_9+c_S\,\delta q_{11}}{2\,c_S}$	$\frac{-\delta q_1+c_S\,\delta q_6+c_S\,\delta q_{10}}{2\,c_S}$	$\frac{\delta q_2+c_S\,\delta q_9+c_S\,\delta q_{11}}{2\,c_S}$	$\frac{\delta q_1+c_S\,\delta q_6+c_S\,\delta q_{10}}{2\,c_S}$	$\frac{-\lambda\,\delta q_3-2\,\mu\,\delta q_3+\lambda\,c_P\,\delta q_4+\lambda}{2\,(\lambda+}$

In[ ]:=
```
DumpSave["lcW_LE.mx", {λW, W, cW}];
```

Isotropic soft solid shear modes

In[ ]:=
```
fSS = f[σ𝔽SS, vel]; TableForm[fSS, TableDirections → Row]
```

Out[ ]//TableForm=

⋯ 1 ⋯

large output | show less | show more | show all | set size limit...

In[ ]:=
```
Dimensions[fSS]
```

Out[ ]= {3, 12}

In[ ]:=
```
fSS[[1, 1]]
```

Out[ ]=

$$
\begin{aligned}
&\frac{1}{4}\Big(14\,\mathcal{D}\,(-1+F_{11})^6+2\,\mathcal{D}\,(-1+F_{11})^7+2\,\mathcal{D}\,F_{12}^6+\mathcal{A}\,F_{13}^2+4\,\mu\,F_{13}^2+\mathcal{A}\,F_{13}^4+4\,\mathcal{D}\,F_{13}^4+2\,\mathcal{D}\,F_{13}^6+\mathcal{A}\,F_{21}^2+\\
&4\,\mu\,F_{21}^2+\mathcal{A}\,F_{13}^2\,F_{21}^2+8\,\mathcal{D}\,F_{13}^2\,F_{21}^2+2\,\mathcal{D}\,F_{13}^4\,F_{21}^2+\mathcal{A}\,F_{21}^4+4\,\mathcal{D}\,F_{21}^4+2\,\mathcal{D}\,F_{13}^2\,F_{21}^4+2\,\mathcal{D}\,F_{21}^6+\\
&2\,\mathcal{A}\,F_{21}^2\,(-1+F_{22})+8\,\mathcal{D}\,F_{13}^2\,F_{21}^2\,(-1+F_{22})+8\,\mathcal{D}\,F_{21}^4\,(-1+F_{22})+8\,\mathcal{D}\,F_{13}^2\,(-1+F_{22})^2+\\
&\mathcal{A}\,F_{21}^2\,(-1+F_{22})^2+8\,\mathcal{D}\,F_{21}^2\,(-1+F_{22})^2+4\,\mathcal{D}\,F_{13}^2\,F_{21}^2\,(-1+F_{22})^2+4\,\mathcal{D}\,F_{21}^4\,(-1+F_{22})^2+\\
&8\,\mathcal{D}\,F_{13}^2\,(-1+F_{22})^3+8\,\mathcal{D}\,F_{21}^2\,(-1+F_{22})^3+2\,\mathcal{D}\,F_{13}^2\,(-1+F_{22})^4+2\,\mathcal{D}\,F_{21}^2\,(-1+F_{22})^4+\\
&3\,\mathcal{A}\,F_{13}\,F_{21}\,F_{23}+4\,\mu\,F_{13}\,F_{21}\,F_{23}+\mathcal{A}\,F_{13}^3\,F_{21}\,F_{23}+12\,\mathcal{D}\,F_{13}^3\,F_{21}\,F_{23}+2\,\mathcal{D}\,F_{13}^5\,F_{21}\,F_{23}+\\
&\mathcal{A}\,F_{13}\,F_{21}^3\,F_{23}+12\,\mathcal{D}\,F_{13}\,F_{21}^3\,F_{23}+2\,\mathcal{D}\,F_{13}\,F_{21}^5\,F_{23}+2\,\mathcal{A}\,F_{13}\,F_{21}\,(-1+F_{22})\,F_{23}+\\
&8\,\mathcal{D}\,F_{13}\,F_{21}^3\,(-1+F_{22})\,F_{23}+\mathcal{A}\,F_{13}\,F_{21}\,(-1+F_{22})^2\,F_{23}+8\,\mathcal{D}\,F_{13}\,F_{21}\,(-1+F_{22})^2\,F_{23}+\\
&4\,\mathcal{D}\,F_{13}\,F_{21}^3\,(-1+F_{22})^2\,F_{23}+8\,\mathcal{D}\,F_{13}\,F_{21}\,(-1+F_{22})^3\,F_{23}+2\,\mathcal{D}\,F_{13}\,F_{21}\,(-1+F_{22})^4\,F_{23}+\\
&\mathcal{A}\,F_{13}^2\,F_{23}^2+4\,\mathcal{D}\,F_{13}^2\,F_{23}^2+4\,\mathcal{D}\,F_{13}^4\,F_{23}^2+\mathcal{A}\,F_{21}^2\,F_{23}^2+4\,\mathcal{D}\,F_{21}^2\,F_{23}^2+16\,\mathcal{D}\,F_{13}^2\,F_{21}^2\,F_{23}^2+4\,\mathcal{D}\,F_{21}^4\,F_{23}^2+\\
&8\,\mathcal{D}\,F_{13}^2\,(-1+F_{22})\,F_{23}^2+8\,\mathcal{D}\,F_{21}^2\,(-1+F_{22})\,F_{23}^2+4\,\mathcal{D}\,F_{13}^2\,(-1+F_{22})^2\,F_{23}^2+4\,\mathcal{D}\,F_{21}^2\,(-1+F_{22})^2\,F_{23}^2+\\
&\mathcal{A}\,F_{13}\,F_{21}\,F_{23}^3+4\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}^3+4\,\mathcal{D}\,F_{13}^3\,F_{21}\,F_{23}^3+4\,\mathcal{D}\,F_{13}\,F_{21}^3\,F_{23}^3+8\,\mathcal{D}\,F_{13}\,F_{21}\,(-1+F_{22})\,F_{23}^3+\\
&4\,\mathcal{D}\,F_{13}\,F_{21}\,(-1+F_{22})^2\,F_{23}^3+2\,\mathcal{D}\,F_{13}^2\,F_{23}^4+2\,\mathcal{D}\,F_{21}^2\,F_{23}^4+2\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}^5+2\,\mathcal{A}\,F_{13}\,F_{31}+\\
&4\,\mu\,F_{13}\,F_{31}+\mathcal{A}\,F_{13}^3\,F_{31}+12\,\mathcal{D}\,F_{13}^3\,F_{31}+2\,\mathcal{D}\,F_{13}^5\,F_{31}+\mathcal{A}\,F_{13}\,F_{21}^2\,F_{31}+12\,\mathcal{D}\,F_{13}\,F_{21}^2\,F_{31}+
\end{aligned}
$$

$$2\,\mathcal{D}\,F_{13}\,F_{21}^{4}\,F_{31}+8\,\mathcal{D}\,F_{13}\,F_{21}^{2}\,(-1+F_{22})\,F_{31}+8\,\mathcal{D}\,F_{13}\,(-1+F_{22})^{2}\,F_{31}+4\,\mathcal{D}\,F_{13}\,F_{21}^{2}\,(-1+F_{22})^{2}\,F_{31}+$$
$$8\,\mathcal{D}\,F_{13}\,(-1+F_{22})^{3}\,F_{31}+2\,\mathcal{D}\,F_{13}\,(-1+F_{22})^{4}\,F_{31}+2\,\mathcal{A}\,F_{21}\,F_{23}\,F_{31}+24\,\mathcal{D}\,F_{13}^{2}\,F_{21}\,F_{23}\,F_{31}+$$
$$8\,\mathcal{D}\,F_{21}^{3}\,F_{23}\,F_{31}+\mathcal{A}\,F_{13}\,F_{23}^{2}\,F_{31}+4\,\mathcal{D}\,F_{13}\,F_{23}^{2}\,F_{31}+4\,\mathcal{D}\,F_{13}^{3}\,F_{23}^{2}\,F_{31}+12\,\mathcal{D}\,F_{13}\,F_{21}^{2}\,F_{23}^{2}\,F_{31}+$$
$$8\,\mathcal{D}\,F_{13}\,(-1+F_{22})\,F_{23}^{2}\,F_{31}+4\,\mathcal{D}\,F_{13}\,(-1+F_{22})^{2}\,F_{23}^{2}\,F_{31}+2\,\mathcal{D}\,F_{13}\,F_{23}^{4}\,F_{31}+\mathcal{A}\,F_{31}^{2}+4\,\mu\,F_{31}^{2}+$$
$$\mathcal{A}\,F_{13}^{2}\,F_{31}^{2}+16\,\mathcal{D}\,F_{13}^{2}\,F_{31}^{2}+2\,\mathcal{D}\,F_{13}^{4}\,F_{31}^{2}+2\,\mathcal{A}\,F_{21}^{2}\,F_{31}^{2}+8\,\mathcal{D}\,F_{21}^{2}\,F_{31}^{2}+4\,\mathcal{D}\,F_{13}^{2}\,F_{21}^{2}\,F_{31}^{2}+$$
$$6\,\mathcal{D}\,F_{21}^{4}\,F_{31}^{2}+8\,\mathcal{D}\,F_{21}^{2}\,(-1+F_{22})\,F_{31}^{2}+8\,\mathcal{D}\,(-1+F_{22})^{2}\,F_{31}^{2}+4\,\mathcal{D}\,F_{21}^{2}\,(-1+F_{22})^{2}\,F_{31}^{2}+$$
$$8\,\mathcal{D}\,(-1+F_{22})^{3}\,F_{31}^{2}+2\,\mathcal{D}\,(-1+F_{22})^{4}\,F_{31}^{2}+\mathcal{A}\,F_{13}\,F_{21}\,F_{23}\,F_{31}^{2}+20\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}\,F_{31}^{2}+$$
$$4\,\mathcal{D}\,F_{13}\,F_{21}^{3}\,F_{23}\,F_{31}^{2}+4\,\mathcal{D}\,F_{23}^{2}\,F_{31}^{2}+4\,\mathcal{D}\,F_{13}^{2}\,F_{23}^{2}\,F_{31}^{2}+4\,\mathcal{D}\,F_{21}^{2}\,F_{23}^{2}\,F_{31}^{2}+8\,\mathcal{D}\,(-1+F_{22})\,F_{23}^{2}\,F_{31}^{2}+$$
$$4\,\mathcal{D}\,(-1+F_{22})^{2}\,F_{23}^{2}\,F_{31}^{2}+2\,\mathcal{D}\,F_{23}^{4}\,F_{31}^{2}+\mathcal{A}\,F_{13}\,F_{31}^{3}+12\,\mathcal{D}\,F_{13}\,F_{31}^{3}+4\,\mathcal{D}\,F_{13}\,F_{21}^{2}\,F_{31}^{3}+$$
$$8\,\mathcal{D}\,F_{21}\,F_{23}\,F_{31}^{3}+\mathcal{A}\,F_{31}^{4}+4\,\mathcal{D}\,F_{31}^{4}+2\,\mathcal{D}\,F_{13}^{2}\,F_{31}^{4}+6\,\mathcal{D}\,F_{21}^{2}\,F_{31}^{4}+2\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}\,F_{31}^{4}+2\,\mathcal{D}\,F_{13}\,F_{31}^{5}+$$
$$2\,\mathcal{D}\,F_{31}^{6}+(-1+F_{11})^{5}\left(\mathcal{A}+36\,\mathcal{D}+6\,\mathcal{D}\,F_{12}^{2}+6\,\mathcal{D}\,F_{13}^{2}+6\,\mathcal{D}\,F_{21}^{2}+6\,\mathcal{D}\,F_{31}^{2}\right)+\mathcal{A}\,F_{13}\,F_{21}\,F_{32}+$$
$$\mathcal{A}\,F_{13}\,F_{21}\,(-1+F_{22})\,F_{32}+8\,\mathcal{D}\,F_{13}^{2}\,F_{23}\,F_{32}+8\,\mathcal{D}\,F_{21}^{2}\,F_{23}\,F_{32}+8\,\mathcal{D}\,F_{13}^{2}\,(-1+F_{22})\,F_{23}\,F_{32}+$$
$$8\,\mathcal{D}\,F_{21}^{2}\,(-1+F_{22})\,F_{23}\,F_{32}+8\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}^{2}\,F_{32}+8\,\mathcal{D}\,F_{13}\,F_{21}\,(-1+F_{22})\,F_{23}^{2}\,F_{32}+2\,\mathcal{A}\,F_{21}\,F_{31}\,F_{32}+$$
$$8\,\mathcal{D}\,F_{13}^{2}\,F_{21}\,F_{31}\,F_{32}+8\,\mathcal{D}\,F_{21}^{3}\,F_{31}\,F_{32}+2\,\mathcal{A}\,F_{21}\,(-1+F_{22})\,F_{31}\,F_{32}+8\,\mathcal{D}\,F_{13}^{2}\,F_{21}\,(-1+F_{22})\,F_{31}\,F_{32}+$$
$$8\,\mathcal{D}\,F_{21}^{3}\,(-1+F_{22})\,F_{31}\,F_{32}+\mathcal{A}\,F_{13}\,F_{23}\,F_{31}\,F_{32}+8\,\mathcal{D}\,F_{13}\,F_{23}\,F_{31}\,F_{32}+8\,\mathcal{D}\,F_{13}\,F_{21}^{2}\,F_{23}\,F_{31}\,F_{32}+$$
$$\mathcal{A}\,F_{13}\,(-1+F_{22})\,F_{23}\,F_{31}\,F_{32}+8\,\mathcal{D}\,F_{13}\,(-1+F_{22})\,F_{23}\,F_{31}\,F_{32}+8\,\mathcal{D}\,F_{13}\,F_{21}^{2}\,(-1+F_{22})\,F_{23}\,F_{31}\,F_{32}+$$
$$8\,\mathcal{D}\,F_{13}\,F_{21}\,F_{31}^{2}\,F_{32}+8\,\mathcal{D}\,F_{13}\,F_{21}\,(-1+F_{22})\,F_{31}^{2}\,F_{32}+8\,\mathcal{D}\,F_{23}\,F_{31}^{2}\,F_{32}+8\,\mathcal{D}\,(-1+F_{22})\,F_{23}\,F_{31}^{2}\,F_{32}+$$
$$8\,\mathcal{D}\,F_{21}\,F_{31}^{3}\,F_{32}+8\,\mathcal{D}\,F_{21}\,(-1+F_{22})\,F_{31}^{3}\,F_{32}+4\,\mathcal{D}\,F_{13}^{2}\,F_{32}^{2}+4\,\mathcal{D}\,F_{21}^{2}\,F_{32}^{2}+8\,\mathcal{D}\,F_{13}^{2}\,(-1+F_{22})\,F_{32}^{2}+$$
$$8\,\mathcal{D}\,F_{21}^{2}\,(-1+F_{22})\,F_{32}^{2}+4\,\mathcal{D}\,F_{13}^{2}\,(-1+F_{22})^{2}\,F_{32}^{2}+4\,\mathcal{D}\,F_{21}^{2}\,(-1+F_{22})^{2}\,F_{32}^{2}+4\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}\,F_{32}^{2}+$$
$$8\,\mathcal{D}\,F_{13}\,F_{21}\,(-1+F_{22})\,F_{23}\,F_{32}^{2}+4\,\mathcal{D}\,F_{13}\,F_{21}\,(-1+F_{22})^{2}\,F_{23}\,F_{32}^{2}+\mathcal{A}\,F_{13}\,F_{31}\,F_{32}^{2}+4\,\mathcal{D}\,F_{13}\,F_{31}\,F_{32}^{2}+$$
$$8\,\mathcal{D}\,F_{13}\,(-1+F_{22})\,F_{31}\,F_{32}^{2}+4\,\mathcal{D}\,F_{13}\,(-1+F_{22})^{2}\,F_{31}\,F_{32}^{2}+\mathcal{A}\,F_{31}^{2}\,F_{32}^{2}+4\,\mathcal{D}\,F_{31}^{2}\,F_{32}^{2}+4\,\mathcal{D}\,F_{13}^{2}\,F_{31}^{2}\,F_{32}^{2}+$$
$$4\,\mathcal{D}\,F_{21}^{2}\,F_{31}^{2}\,F_{32}^{2}+8\,\mathcal{D}\,(-1+F_{22})\,F_{31}^{2}\,F_{32}^{2}+4\,\mathcal{D}\,(-1+F_{22})^{2}\,F_{31}^{2}\,F_{32}^{2}+4\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}\,F_{31}^{2}\,F_{32}^{2}+$$
$$4\,\mathcal{D}\,F_{13}\,F_{31}^{3}\,F_{32}^{2}+4\,\mathcal{D}\,F_{31}^{4}\,F_{32}^{2}+2\,\mathcal{D}\,F_{13}^{2}\,F_{32}^{4}+2\,\mathcal{D}\,F_{21}^{2}\,F_{32}^{4}+2\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}\,F_{32}^{4}+2\,\mathcal{D}\,F_{13}\,F_{31}\,F_{32}^{4}+$$
$$2\,\mathcal{D}\,F_{31}^{2}\,F_{32}^{4}+2\,\mathcal{D}\,F_{12}^{5}\,(F_{21}\,F_{22}+F_{31}\,F_{32})+2\,\mathcal{A}\,F_{13}^{2}\,(-1+F_{33})+8\,\mathcal{D}\,F_{13}^{4}\,(-1+F_{33})+$$
$$8\,\mathcal{D}\,F_{13}^{2}\,F_{21}^{2}\,(-1+F_{33})+2\,\mathcal{A}\,F_{13}\,F_{21}\,F_{23}\,(-1+F_{33})+8\,\mathcal{D}\,F_{13}^{3}\,F_{21}\,F_{23}\,(-1+F_{33})+$$
$$8\,\mathcal{D}\,F_{13}^{2}\,F_{23}^{2}\,(-1+F_{33})+8\,\mathcal{D}\,F_{21}^{2}\,F_{23}^{2}\,(-1+F_{33})+8\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}^{3}\,(-1+F_{33})+4\,\mathcal{A}\,F_{13}\,F_{31}\,(-1+F_{33})+$$
$$4\,\mu\,F_{13}\,F_{31}\,(-1+F_{33})+\mathcal{A}\,F_{13}^{3}\,F_{31}\,(-1+F_{33})+20\,\mathcal{D}\,F_{13}^{3}\,F_{31}\,(-1+F_{33})+2\,\mathcal{D}\,F_{13}^{5}\,F_{31}\,(-1+F_{33})+$$
$$\mathcal{A}\,F_{13}\,F_{21}^{2}\,F_{31}\,(-1+F_{33})+12\,\mathcal{D}\,F_{13}\,F_{21}^{2}\,F_{31}\,(-1+F_{33})+2\,\mathcal{D}\,F_{13}\,F_{21}^{4}\,F_{31}\,(-1+F_{33})+$$
$$8\,\mathcal{D}\,F_{13}\,F_{21}^{2}\,(-1+F_{22})\,F_{31}\,(-1+F_{33})+8\,\mathcal{D}\,F_{13}\,(-1+F_{22})^{2}\,F_{31}\,(-1+F_{33})+$$
$$4\,\mathcal{D}\,F_{13}\,F_{21}^{2}\,(-1+F_{22})^{2}\,F_{31}\,(-1+F_{33})+8\,\mathcal{D}\,F_{13}\,(-1+F_{22})^{3}\,F_{31}\,(-1+F_{33})+$$
$$2\,\mathcal{D}\,F_{13}\,(-1+F_{22})^{4}\,F_{31}\,(-1+F_{33})+2\,\mathcal{A}\,F_{21}\,F_{23}\,F_{31}\,(-1+F_{33})+24\,\mathcal{D}\,F_{13}^{2}\,F_{21}\,F_{23}\,F_{31}\,(-1+F_{33})+$$
$$8\,\mathcal{D}\,F_{21}^{3}\,F_{23}\,F_{31}\,(-1+F_{33})+\mathcal{A}\,F_{13}\,F_{23}^{2}\,F_{31}\,(-1+F_{33})+12\,\mathcal{D}\,F_{13}\,F_{23}^{2}\,F_{31}\,(-1+F_{33})+$$
$$4\,\mathcal{D}\,F_{13}^{3}\,F_{23}^{2}\,F_{31}\,(-1+F_{33})+12\,\mathcal{D}\,F_{13}\,F_{21}^{2}\,F_{23}^{2}\,F_{31}\,(-1+F_{33})+8\,\mathcal{D}\,F_{13}\,(-1+F_{22})\,F_{23}^{2}\,F_{31}\,(-1+F_{33})+$$
$$4\,\mathcal{D}\,F_{13}\,(-1+F_{22})^{2}\,F_{23}^{2}\,F_{31}\,(-1+F_{33})+2\,\mathcal{D}\,F_{13}\,F_{23}^{4}\,F_{31}\,(-1+F_{33})+2\,\mathcal{A}\,F_{31}^{2}\,(-1+F_{33})+$$
$$32\,\mathcal{D}\,F_{13}^{2}\,F_{31}^{2}\,(-1+F_{33})+8\,\mathcal{D}\,F_{21}^{2}\,F_{31}^{2}\,(-1+F_{33})+24\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}\,F_{31}^{2}\,(-1+F_{33})+$$
$$8\,\mathcal{D}\,F_{23}^{2}\,F_{31}^{2}\,(-1+F_{33})+\mathcal{A}\,F_{13}\,F_{31}^{3}\,(-1+F_{33})+20\,\mathcal{D}\,F_{13}\,F_{31}^{3}\,(-1+F_{33})+$$
$$4\,\mathcal{D}\,F_{13}\,F_{21}^{2}\,F_{31}^{3}\,(-1+F_{33})+8\,\mathcal{D}\,F_{21}\,F_{23}\,F_{31}^{3}\,(-1+F_{33})+8\,\mathcal{D}\,F_{31}^{4}\,(-1+F_{33})+$$
$$2\,\mathcal{D}\,F_{13}\,F_{31}^{5}\,(-1+F_{33})+\mathcal{A}\,F_{13}\,F_{21}\,F_{32}\,(-1+F_{33})+\mathcal{A}\,F_{13}\,F_{21}\,(-1+F_{22})\,F_{32}\,(-1+F_{33})+$$
$$8\,\mathcal{D}\,F_{13}^{2}\,F_{23}\,F_{32}\,(-1+F_{33})+8\,\mathcal{D}\,F_{21}^{2}\,F_{23}\,F_{32}\,(-1+F_{33})+8\,\mathcal{D}\,F_{13}^{2}\,(-1+F_{22})\,F_{23}\,F_{32}\,(-1+F_{33})+$$
$$8\,\mathcal{D}\,F_{21}^{2}\,(-1+F_{22})\,F_{23}\,F_{32}\,(-1+F_{33})+8\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}^{2}\,F_{32}\,(-1+F_{33})+$$
$$8\,\mathcal{D}\,F_{13}\,F_{21}\,(-1+F_{22})\,F_{23}^{2}\,F_{32}\,(-1+F_{33})+16\,\mathcal{D}\,F_{13}\,F_{23}\,F_{31}\,F_{32}\,(-1+F_{33})+$$

$$16\,\mathcal{D}\,F_{13}\,(-1+F_{22})\,F_{23}\,F_{31}\,F_{32}\,(-1+F_{33})+8\,\mathcal{D}\,F_{13}\,F_{21}\,F_{31}^2\,F_{32}\,(-1+F_{33})+$$
$$8\,\mathcal{D}\,F_{13}\,F_{21}\,(-1+F_{22})\,F_{31}^2\,F_{32}\,(-1+F_{33})+8\,\mathcal{D}\,F_{23}\,F_{31}^2\,F_{32}\,(-1+F_{33})+$$
$$8\,\mathcal{D}\,(-1+F_{22})\,F_{23}\,F_{31}^2\,F_{32}\,(-1+F_{33})+8\,\mathcal{D}\,F_{13}^2\,F_{32}^2\,(-1+F_{33})+8\,\mathcal{D}\,F_{21}^2\,F_{32}^2\,(-1+F_{33})+$$
$$8\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}\,F_{32}^2\,(-1+F_{33})+\mathcal{A}\,F_{13}\,F_{31}\,F_{32}^2\,(-1+F_{33})+12\,\mathcal{D}\,F_{13}\,F_{31}\,F_{32}^2\,(-1+F_{33})+$$
$$8\,\mathcal{D}\,F_{13}\,(-1+F_{22})\,F_{31}\,F_{32}^2\,(-1+F_{33})+4\,\mathcal{D}\,F_{13}\,(-1+F_{22})^2\,F_{31}\,F_{32}^2\,(-1+F_{33})+$$
$$8\,\mathcal{D}\,F_{31}^2\,F_{32}^2\,(-1+F_{33})+4\,\mathcal{D}\,F_{13}\,F_{31}^3\,F_{32}^2\,(-1+F_{33})+2\,\mathcal{D}\,F_{13}\,F_{31}\,F_{32}^4\,(-1+F_{33})+\mathcal{A}\,F_{13}^2\,(-1+F_{33})^2+$$
$$8\,\mathcal{D}\,F_{13}^2\,(-1+F_{33})^2+4\,\mathcal{D}\,F_{13}^4\,(-1+F_{33})^2+8\,\mathcal{D}\,F_{21}^2\,(-1+F_{33})^2+4\,\mathcal{D}\,F_{13}^2\,F_{21}^2\,(-1+F_{33})^2+$$
$$\mathcal{A}\,F_{13}\,F_{21}\,F_{23}\,(-1+F_{33})^2+8\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}\,(-1+F_{33})^2+4\,\mathcal{D}\,F_{13}^3\,F_{21}\,F_{23}\,(-1+F_{33})^2+$$
$$4\,\mathcal{D}\,F_{13}^2\,F_{23}^2\,(-1+F_{33})^2+4\,\mathcal{D}\,F_{21}^2\,F_{23}^2\,(-1+F_{33})^2+4\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}^3\,(-1+F_{33})^2+$$
$$3\,\mathcal{A}\,F_{13}\,F_{31}\,(-1+F_{33})^2+8\,\mathcal{D}\,F_{13}\,F_{31}\,(-1+F_{33})^2+12\,\mathcal{D}\,F_{13}^3\,F_{31}\,(-1+F_{33})^2+$$
$$12\,\mathcal{D}\,F_{13}\,F_{23}^2\,F_{31}\,(-1+F_{33})^2+\mathcal{A}\,F_{31}^2\,(-1+F_{33})^2+8\,\mathcal{D}\,F_{31}^2\,(-1+F_{33})^2+16\,\mathcal{D}\,F_{13}^2\,F_{31}^2\,(-1+F_{33})^2+$$
$$4\,\mathcal{D}\,F_{21}^2\,F_{31}^2\,(-1+F_{33})^2+12\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}\,F_{31}^2\,(-1+F_{33})^2+4\,\mathcal{D}\,F_{23}^2\,F_{31}^2\,(-1+F_{33})^2+$$
$$12\,\mathcal{D}\,F_{13}\,F_{31}^3\,(-1+F_{33})^2+4\,\mathcal{D}\,F_{31}^4\,(-1+F_{33})^2+8\,\mathcal{D}\,F_{13}\,F_{23}\,F_{31}\,F_{32}\,(-1+F_{33})^2+$$
$$8\,\mathcal{D}\,F_{13}\,(-1+F_{22})\,F_{23}\,F_{31}\,F_{32}\,(-1+F_{33})^2+4\,\mathcal{D}\,F_{13}^2\,F_{32}^2\,(-1+F_{33})^2+4\,\mathcal{D}\,F_{21}^2\,F_{32}^2\,(-1+F_{33})^2+$$
$$4\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}\,F_{32}^2\,(-1+F_{33})^2+12\,\mathcal{D}\,F_{13}\,F_{31}\,F_{32}^2\,(-1+F_{33})^2+4\,\mathcal{D}\,F_{31}^2\,F_{32}^2\,(-1+F_{33})^2+$$
$$8\,\mathcal{D}\,F_{13}^2\,(-1+F_{33})^3+8\,\mathcal{D}\,F_{21}^2\,(-1+F_{33})^3+8\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}\,(-1+F_{33})^3+\mathcal{A}\,F_{13}\,F_{31}\,(-1+F_{33})^3+$$
$$16\,\mathcal{D}\,F_{13}\,F_{31}\,(-1+F_{33})^3+4\,\mathcal{D}\,F_{13}^3\,F_{31}\,(-1+F_{33})^3+4\,\mathcal{D}\,F_{13}\,F_{23}^2\,F_{31}\,(-1+F_{33})^3+$$
$$8\,\mathcal{D}\,F_{31}^2\,(-1+F_{33})^3+4\,\mathcal{D}\,F_{13}\,F_{31}^3\,(-1+F_{33})^3+4\,\mathcal{D}\,F_{13}\,F_{31}\,F_{32}^2\,(-1+F_{33})^3+2\,\mathcal{D}\,F_{13}^2\,(-1+F_{33})^4+$$
$$2\,\mathcal{D}\,F_{21}^2\,(-1+F_{33})^4+2\,\mathcal{D}\,F_{13}\,F_{21}\,F_{23}\,(-1+F_{33})^4+10\,\mathcal{D}\,F_{13}\,F_{31}\,(-1+F_{33})^4+2\,\mathcal{D}\,F_{31}^2\,(-1+F_{33})^4+$$
$$2\,\mathcal{D}\,F_{13}\,F_{31}\,(-1+F_{33})^5+F_{12}^4\,\big(\mathcal{A}+4\,\mathcal{D}+6\,\mathcal{D}\,F_{13}^2+2\,\mathcal{D}\,F_{21}^2+8\,\mathcal{D}\,(-1+F_{22})+4\,\mathcal{D}\,(-1+F_{22})^2+$$
$$2\,\mathcal{D}\,F_{31}^2+4\,\mathcal{D}\,F_{32}^2+2\,\mathcal{D}\,F_{13}\,(F_{21}\,F_{23}+F_{31}\,F_{33})\big)+5\,(-1+F_{11})^4\,\big(\mathcal{A}+8\,\mathcal{D}+6\,\mathcal{D}\,F_{12}^2+$$
$$6\,\mathcal{D}\,F_{13}^2+6\,\mathcal{D}\,F_{21}^2+6\,\mathcal{D}\,F_{31}^2+2\,\mathcal{D}\,F_{12}\,(F_{21}\,F_{22}+F_{31}\,F_{32})+2\,\mathcal{D}\,F_{13}\,(F_{21}\,F_{23}+F_{31}\,F_{33})\big)+$$
$$F_{12}^3\,\big(4\,\mathcal{D}\,F_{13}^2\,F_{31}\,F_{32}+F_{31}\,F_{32}\,\big(\mathcal{A}+12\,\mathcal{D}+8\,\mathcal{D}\,(-1+F_{22})+4\,\mathcal{D}\,(-1+F_{22})^2+4\,\mathcal{D}\,F_{32}^2\big)+$$
$$F_{21}\,F_{22}\,\big(\mathcal{A}+12\,\mathcal{D}+4\,\mathcal{D}\,F_{13}^2+8\,\mathcal{D}\,(-1+F_{22})+4\,\mathcal{D}\,(-1+F_{22})^2+4\,\mathcal{D}\,F_{32}^2\big)+$$
$$8\,\mathcal{D}\,F_{13}\,(F_{22}\,F_{23}+F_{32}\,F_{33})\big)+2\,(-1+F_{11})^3$$
$$\big(4\,\mathcal{A}+8\,\mathcal{D}+2\,\mu+3\,\mathcal{D}\,F_{12}^4+3\,\mathcal{D}\,F_{13}^4+\mathcal{A}\,F_{21}^2+26\,\mathcal{D}\,F_{21}^2+3\,\mathcal{D}\,F_{21}^4+4\,\mathcal{D}\,F_{21}^2\,(-1+F_{22})+$$
$$4\,\mathcal{D}\,(-1+F_{22})^2+2\,\mathcal{D}\,F_{21}^2\,(-1+F_{22})^2+4\,\mathcal{D}\,(-1+F_{22})^3+\mathcal{D}\,(-1+F_{22})^4+2\,\mathcal{D}\,F_{23}^2+2\,\mathcal{D}\,F_{21}^2\,F_{23}^2+$$
$$4\,\mathcal{D}\,(-1+F_{22})\,F_{23}^2+2\,\mathcal{D}\,(-1+F_{22})^2\,F_{23}^2+\mathcal{D}\,F_{23}^4+4\,\mathcal{D}\,F_{21}\,F_{23}\,F_{31}+\mathcal{A}\,F_{31}^2+26\,\mathcal{D}\,F_{31}^2+$$
$$6\,\mathcal{D}\,F_{21}^2\,F_{31}^2+3\,\mathcal{D}\,F_{31}^4+4\,\mathcal{D}\,F_{23}\,F_{32}+4\,\mathcal{D}\,(-1+F_{22})\,F_{23}\,F_{32}+4\,\mathcal{D}\,F_{21}\,F_{31}\,F_{32}+4\,\mathcal{D}\,F_{21}$$
$$(-1+F_{22})\,F_{31}\,F_{32}+2\,\mathcal{D}\,F_{32}^2+4\,\mathcal{D}\,(-1+F_{22})\,F_{32}^2+2\,\mathcal{D}\,(-1+F_{22})^2\,F_{32}^2+2\,\mathcal{D}\,F_{31}^2\,F_{32}^2+\mathcal{D}\,F_{32}^4+$$
$$F_{12}^2\,\big(\mathcal{A}+26\,\mathcal{D}+6\,\mathcal{D}\,F_{13}^2+4\,\mathcal{D}\,F_{21}^2+4\,\mathcal{D}\,(-1+F_{22})+2\,\mathcal{D}\,(-1+F_{22})^2+4\,\mathcal{D}\,F_{31}^2+2\,\mathcal{D}\,F_{32}^2\big)+$$
$$F_{13}^2\,\big(\mathcal{A}+26\,\mathcal{D}+4\,\mathcal{D}\,F_{21}^2+2\,\mathcal{D}\,F_{23}^2+4\,\mathcal{D}\,F_{31}^2+4\,\mathcal{D}\,(-1+F_{33})+2\,\mathcal{D}\,(-1+F_{33})^2\big)+$$
$$4\,\mathcal{D}\,F_{23}^2\,(-1+F_{33})+4\,\mathcal{D}\,F_{21}\,F_{23}\,F_{31}\,(-1+F_{33})+4\,\mathcal{D}\,F_{31}^2\,(-1+F_{33})+4\,\mathcal{D}\,F_{23}\,F_{32}\,(-1+F_{33})+$$
$$4\,\mathcal{D}\,(-1+F_{22})\,F_{23}\,F_{32}\,(-1+F_{33})+4\,\mathcal{D}\,F_{32}^2\,(-1+F_{33})+4\,\mathcal{D}\,(-1+F_{33})^2+2\,\mathcal{D}\,F_{23}^2\,(-1+F_{33})^2+$$
$$2\,\mathcal{D}\,F_{31}^2\,(-1+F_{33})^2+2\,\mathcal{D}\,F_{32}^2\,(-1+F_{33})^2+4\,\mathcal{D}\,(-1+F_{33})^3+\mathcal{D}\,(-1+F_{33})^4+$$
$$20\,\mathcal{D}\,F_{13}\,(F_{21}\,F_{23}+F_{31}\,F_{33})+4\,\mathcal{D}\,F_{12}\,(5\,F_{21}\,F_{22}+5\,F_{31}\,F_{32}+F_{13}\,(F_{22}\,F_{23}+F_{32}\,F_{33}))\big)+$$
$$(-1+F_{11})^2\,\big(18\,\mathcal{D}\,F_{12}^4+18\,\mathcal{D}\,F_{13}^4+12\,\mathcal{D}\,F_{12}^3\,(F_{21}\,F_{22}+F_{31}\,F_{32})+6\,F_{13}^2\,\big(\mathcal{A}+6\,\mathcal{D}+4\,\mathcal{D}\,F_{21}^2+$$
$$2\,\mathcal{D}\,F_{23}^2+4\,\mathcal{D}\,F_{31}^2+4\,\mathcal{D}\,(-1+F_{33})+2\,\mathcal{D}\,(-1+F_{33})^2\big)+12\,\mathcal{D}\,F_{13}^3\,(F_{21}\,F_{23}+F_{31}\,F_{33})+$$
$$3\,F_{13}\,\big(\mathcal{A}+16\,\mathcal{D}+4\,\mathcal{D}\,F_{21}^2+4\,\mathcal{D}\,F_{31}^2\big)\,(F_{21}\,F_{23}+F_{31}\,F_{33})+6\,F_{12}^2\,\big(\mathcal{A}+6\,\mathcal{D}+6\,\mathcal{D}\,F_{13}^2+4\,\mathcal{D}\,F_{21}^2+$$
$$4\,\mathcal{D}\,(-1+F_{22})+2\,\mathcal{D}\,(-1+F_{22})^2+4\,\mathcal{D}\,F_{31}^2+2\,\mathcal{D}\,F_{32}^2+2\,\mathcal{D}\,F_{13}\,(F_{21}\,F_{23}+F_{31}\,F_{33})\big)+$$

$$3 F_{12} (4 \mathcal{D} F_{21}^3 F_{22} + F_{21} F_{22} (\mathcal{A} + 16 \mathcal{D} + 4 \mathcal{D} F_{13}^2 + 4 \mathcal{D} F_{31}^2) + 4 \mathcal{D} F_{13}^2 F_{31} F_{32} + 4 \mathcal{D} F_{21}^2 F_{31} F_{32} +$$

$$F_{31} (\mathcal{A} + 16 \mathcal{D} + 4 \mathcal{D} F_{31}^2) F_{32} + 8 \mathcal{D} F_{13} (F_{22} F_{23} + F_{32} F_{33})) + 2 (2 \mathcal{A} + 6 \mu + 9 \mathcal{D} F_{21}^4 +$$

$$12 \mathcal{D} (-1 + F_{22})^3 + 3 \mathcal{D} (-1 + F_{22})^4 + 6 \mathcal{D} F_{23}^2 + 3 \mathcal{D} F_{23}^4 + 3 \mathcal{A} F_{31}^2 + 18 \mathcal{D} F_{31}^2 + 9 \mathcal{D} F_{31}^4 +$$

$$3 F_{21}^2 (\mathcal{A} + 6 \mathcal{D} + 4 \mathcal{D} (-1 + F_{22}) + 2 \mathcal{D} (-1 + F_{22})^2 + 2 \mathcal{D} F_{23}^2 + 6 \mathcal{D} F_{31}^2) + 12 \mathcal{D} F_{23} F_{32} +$$

$$6 \mathcal{D} F_{32}^2 + 6 \mathcal{D} F_{31}^2 F_{32}^2 + 3 \mathcal{D} F_{32}^4 + 6 \mathcal{D} (-1 + F_{22})^2 (2 + F_{23}^2 + F_{32}^2) + 12 \mathcal{D} F_{23}^2 (-1 + F_{33}) +$$

$$12 \mathcal{D} F_{31}^2 (-1 + F_{33}) + 12 \mathcal{D} F_{23} F_{32} (-1 + F_{33}) + 12 \mathcal{D} F_{32}^2 (-1 + F_{33}) + 12 \mathcal{D} (-1 + F_{33})^2 +$$

$$6 \mathcal{D} F_{23}^2 (-1 + F_{33})^2 + 6 \mathcal{D} F_{31}^2 (-1 + F_{33})^2 + 6 \mathcal{D} F_{32}^2 (-1 + F_{33})^2 + 12 \mathcal{D} (-1 + F_{33})^3 + 3 \mathcal{D}$$

$$(-1 + F_{33})^4 + 12 \mathcal{D} F_{21} F_{31} (F_{22} F_{32} + F_{23} F_{33}) + 12 \mathcal{D} (-1 + F_{22}) (F_{23}^2 + F_{32}^2 + F_{23} F_{32} F_{33}))) +$$

$$F_{12} (2 \mathcal{D} F_{21}^5 F_{22} + F_{21}^3 F_{22} (\mathcal{A} + 12 \mathcal{D} + 8 \mathcal{D} (-1 + F_{22}) + 4 \mathcal{D} (-1 + F_{22})^2 + 4 \mathcal{D} F_{23}^2 + 4 \mathcal{D} F_{31}^2) +$$

$$2 \mathcal{D} F_{13}^4 F_{31} F_{32} + 2 \mathcal{D} F_{21}^4 F_{31} F_{32} +$$

$$F_{13}^2 F_{31} (4 \mathcal{D} F_{23}^2 F_{32} + F_{32} (\mathcal{A} + 20 \mathcal{D} + 24 \mathcal{D} (-1 + F_{33}) + 12 \mathcal{D} (-1 + F_{33})^2) + 8 \mathcal{D} F_{22} F_{23} F_{33}) +$$

$$8 \mathcal{D} F_{13}^3 (F_{22} F_{23} + F_{32} F_{33}) + 2 F_{13} (F_{22} F_{23} (\mathcal{A} + 4 \mathcal{D} F_{31}^2) + (\mathcal{A} + 12 \mathcal{D} F_{31}^2) F_{32} F_{33}) +$$

$$F_{21}^2 (F_{31} (4 \mathcal{D} F_{23}^2 F_{32} + (\mathcal{A} + 20 \mathcal{D} + 24 \mathcal{D} (-1 + F_{22}) + 12 \mathcal{D} (-1 + F_{22})^2 + 4 \mathcal{D} F_{31}^2) F_{32} +$$

$$8 \mathcal{D} F_{22} F_{23} F_{33}) + 8 \mathcal{D} F_{13} (3 F_{22} F_{23} + F_{32} F_{33})) +$$

$$F_{21} (2 \mathcal{A} + 4 \mu + 10 \mathcal{D} (-1 + F_{22})^4 + 2 \mathcal{D} (-1 + F_{22})^5 + 2 \mathcal{D} F_{13}^4 F_{22} + \mathcal{A} F_{23}^2 + 4 \mathcal{D} F_{23}^2 +$$

$$2 \mathcal{D} F_{23}^4 + \mathcal{A} F_{31}^2 + 12 \mathcal{D} F_{31}^2 + 2 \mathcal{D} F_{31}^4 + \mathcal{A} F_{23} F_{32} + 8 \mathcal{D} F_{23} F_{32} + 8 \mathcal{D} F_{23} F_{31}^2 F_{32} +$$

$$\mathcal{A} F_{32}^2 + 4 \mathcal{D} F_{32}^2 + 12 \mathcal{D} F_{31}^2 F_{32}^2 + 2 \mathcal{D} F_{32}^4 + (-1 + F_{22})^3 (\mathcal{A} + 16 \mathcal{D} + 4 \mathcal{D} F_{23}^2 + 4 \mathcal{D} F_{32}^2) +$$

$$8 \mathcal{D} F_{23}^2 (-1 + F_{33}) + 8 \mathcal{D} F_{31}^2 (-1 + F_{33}) + \mathcal{A} F_{23} F_{32} (-1 + F_{33}) + 8 \mathcal{D} F_{23} F_{32} (-1 + F_{33}) +$$

$$8 \mathcal{D} F_{23} F_{31}^2 F_{32} (-1 + F_{33}) + 8 \mathcal{D} F_{32}^2 (-1 + F_{33}) + 8 \mathcal{D} (-1 + F_{33})^2 + 4 \mathcal{D} F_{23}^2 (-1 + F_{33})^2 +$$

$$4 \mathcal{D} F_{31}^2 (-1 + F_{33})^2 + 4 \mathcal{D} F_{32}^2 (-1 + F_{33})^2 + 8 \mathcal{D} (-1 + F_{33})^3 + 2 \mathcal{D} (-1 + F_{33})^4 +$$

$$16 \mathcal{D} F_{13} F_{31} (F_{23} F_{32} + F_{33} + (-1 + F_{22}) F_{33}) + (-1 + F_{22})^2 (3 \mathcal{A} + 8 \mathcal{D} + 12 \mathcal{D} F_{23}^2 + 12 \mathcal{D} F_{32}^2 +$$

$$8 \mathcal{D} F_{23} F_{32} F_{33}) + F_{13}^2 (\mathcal{A} + 12 \mathcal{D} + 12 \mathcal{D} F_{23}^2 + (-1 + F_{22}) (\mathcal{A} + 12 \mathcal{D} + 12 \mathcal{D} F_{23}^2 + 8 \mathcal{D}$$

$$(-1 + F_{33}) + 4 \mathcal{D} (-1 + F_{33})^2) + 8 \mathcal{D} (-1 + F_{33}) + 4 \mathcal{D} (-1 + F_{33})^2 + 8 \mathcal{D} F_{23} F_{32} F_{33}) +$$

$$(-1 + F_{22}) (4 \mathcal{A} + 4 \mu + 2 \mathcal{D} F_{23}^4 + 2 \mathcal{D} F_{31}^4 + \mathcal{A} F_{32}^2 + 12 \mathcal{D} F_{32}^2 + 2 \mathcal{D} F_{32}^4 +$$

$$F_{23}^2 (\mathcal{A} + 12 \mathcal{D} + 8 \mathcal{D} (-1 + F_{33}) + 4 \mathcal{D} (-1 + F_{33})^2) + F_{31}^2 (\mathcal{A} + 12 \mathcal{D} + 12 \mathcal{D} F_{32}^2 + 8$$

$$\mathcal{D} (-1 + F_{33}) + 4 \mathcal{D} (-1 + F_{33})^2) + 8 \mathcal{D} F_{32}^2 (-1 + F_{33}) + 8 \mathcal{D} (-1 + F_{33})^2 +$$

$$4 \mathcal{D} F_{32}^2 (-1 + F_{33})^2 + 8 \mathcal{D} (-1 + F_{33})^3 + 2 \mathcal{D} (-1 + F_{33})^4 + 16 \mathcal{D} F_{23} F_{32} F_{33})) +$$

$$F_{31} (2 \mathcal{D} F_{23}^4 F_{32} + F_{32} (3 \mathcal{A} + 4 \mu + 8 \mathcal{D} (-1 + F_{22})^3 + 2 \mathcal{D} (-1 + F_{22})^4 + 2 \mathcal{D} F_{31}^4 + \mathcal{A} F_{32}^2 +$$

$$4 \mathcal{D} F_{32}^2 + 2 \mathcal{D} F_{32}^4 + 2 (-1 + F_{22}) (\mathcal{A} + 4 \mathcal{D} F_{32}^2) + (-1 + F_{22})^2 (\mathcal{A} + 8 \mathcal{D} + 4 \mathcal{D} F_{32}^2) +$$

$$F_{31}^2 (\mathcal{A} + 12 \mathcal{D} + 4 \mathcal{D} F_{32}^2 + 8 \mathcal{D} (-1 + F_{33}) + 4 \mathcal{D} (-1 + F_{33})^2) +$$

$$2 \mathcal{A} (-1 + F_{33}) + 8 \mathcal{D} F_{32}^2 (-1 + F_{33}) + \mathcal{A} (-1 + F_{33})^2 + 8 \mathcal{D} (-1 + F_{33})^2 +$$

$$4 \mathcal{D} F_{32}^2 (-1 + F_{33})^2 + 8 \mathcal{D} (-1 + F_{33})^3 + 2 \mathcal{D} (-1 + F_{33})^4) +$$

$$F_{22} F_{23} (\mathcal{A} + 8 \mathcal{D} F_{32}^2) F_{33} + 4 \mathcal{D} F_{23}^2 F_{32} (2 (-1 + F_{22}) + (-1 + F_{22})^2 + F_{33}^2))) +$$

$$F_{12}^2 (\mathcal{A} + 4 \mu + 6 \mathcal{D} F_{13}^4 + 2 \mathcal{D} F_{21}^4 + 2 \mathcal{A} (-1 + F_{22}) + \mathcal{A} (-1 + F_{22})^2 + 8 \mathcal{D} (-1 + F_{22})^2 +$$

$$8 \mathcal{D} (-1 + F_{22})^3 + 2 \mathcal{D} (-1 + F_{22})^4 + 4 \mathcal{D} F_{23}^2 + 8 \mathcal{D} (-1 + F_{22}) F_{23}^2 +$$

$$4 \mathcal{D} (-1 + F_{22})^2 F_{23}^2 + 2 \mathcal{D} F_{23}^4 + \mathcal{A} F_{31}^2 + 8 \mathcal{D} F_{31}^2 + 8 \mathcal{D} (-1 + F_{22}) F_{31}^2 +$$

$$4 \mathcal{D} (-1 + F_{22})^2 F_{31}^2 + 2 \mathcal{D} F_{31}^4 + 8 \mathcal{D} F_{23} F_{32} + 8 \mathcal{D} (-1 + F_{22}) F_{23} F_{32} + \mathcal{A} F_{32}^2 +$$

$$4 \mathcal{D} F_{32}^2 + 8 \mathcal{D} (-1 + F_{22}) F_{32}^2 + 4 \mathcal{D} (-1 + F_{22})^2 F_{32}^2 + 16 \mathcal{D} F_{31}^2 F_{32}^2 + 2 \mathcal{D} F_{32}^4 +$$

$$F_{21}^2 (\mathcal{A} + 16 \mathcal{D} + 32 \mathcal{D} (-1 + F_{22}) + 16 \mathcal{D} (-1 + F_{22})^2 + 4 \mathcal{D} F_{23}^2 + 4 \mathcal{D} F_{31}^2 + 4 \mathcal{D} F_{32}^2) +$$

$$2 F_{13}^2 \left(\mathcal{A} + 4 \mathcal{D} + 2 \mathcal{D} F_{21}^2 + 4 \mathcal{D} (-1 + F_{22}) + 2 \mathcal{D} (-1 + F_{22})^2 + 2 \mathcal{D} F_{23}^2 + 2 \mathcal{D} F_{31}^2 +\right.$$

$$\left.2 \mathcal{D} F_{32}^2 + 4 \mathcal{D} (-1 + F_{33}) + 2 \mathcal{D} (-1 + F_{33})^2\right) + 8 \mathcal{D} F_{23}^2 (-1 + F_{33}) + 8 \mathcal{D} F_{31}^2 (-1 + F_{33}) +$$

$$8 \mathcal{D} F_{23} F_{32} (-1 + F_{33}) + 8 \mathcal{D} (-1 + F_{22}) F_{23} F_{32} (-1 + F_{33}) + 8 \mathcal{D} F_{32}^2 (-1 + F_{33}) + 8 \mathcal{D} (-1 + F_{33})^2 +$$

$$4 \mathcal{D} F_{23}^2 (-1 + F_{33})^2 + 4 \mathcal{D} F_{31}^2 (-1 + F_{33})^2 + 4 \mathcal{D} F_{32}^2 (-1 + F_{33})^2 + 8 \mathcal{D} (-1 + F_{33})^3 +$$

$$2 \mathcal{D} (-1 + F_{33})^4 + 8 \mathcal{D} F_{21} F_{31} (3 F_{22} F_{32} + F_{23} F_{33}) + 4 \mathcal{D} F_{13}^3 (F_{21} F_{23} + F_{31} F_{33}) +$$

$$F_{13} \left(F_{21} \left(F_{23} \left(\mathcal{A} + 20 \mathcal{D} + 24 \mathcal{D} (-1 + F_{22}) + 12 \mathcal{D} (-1 + F_{22})^2 + 4 \mathcal{D} F_{32}^2\right) + 8 \mathcal{D} F_{22} F_{32} F_{33}\right) +\right.$$

$$F_{31} \left(8 \mathcal{D} F_{23} F_{32} + 4 \mathcal{D} (-1 + F_{22})^2 F_{33} +\right.$$

$$\left.\left.\left(\mathcal{A} + 12 \mathcal{D} + 12 \mathcal{D} F_{32}^2\right) F_{33} + 8 \mathcal{D} (-1 + F_{22}) (F_{23} F_{32} + F_{33})\right)\right)\Big)\Big) +$$

$$(-1 + F_{11}) \left(8 \mu + 2 \mathcal{D} F_{12}^6 + 2 \mathcal{D} F_{13}^6 + 5 \mathcal{A} F_{21}^2 + 8 \mathcal{D} F_{21}^2 + 4 \mu F_{21}^2 + \mathcal{A} F_{21}^4 + 16 \mathcal{D} F_{21}^4 +\right.$$

$$2 \mathcal{D} F_{21}^6 + 2 \mathcal{A} F_{21}^2 (-1 + F_{22}) + 16 \mathcal{D} F_{21}^2 (-1 + F_{22}) + 8 \mathcal{D} F_{21}^4 (-1 + F_{22}) + 16 \mathcal{D} (-1 + F_{22})^2 +$$

$$\mathcal{A} F_{21}^2 (-1 + F_{22})^2 + 16 \mathcal{D} F_{21}^2 (-1 + F_{22})^2 + 4 \mathcal{D} F_{21}^4 (-1 + F_{22})^2 + 16 \mathcal{D} (-1 + F_{22})^3 +$$

$$8 \mathcal{D} F_{21}^2 (-1 + F_{22})^3 + 4 \mathcal{D} (-1 + F_{22})^4 + 2 \mathcal{D} F_{21}^2 (-1 + F_{22})^4 + 8 \mathcal{D} F_{23}^2 + \mathcal{A} F_{21}^2 F_{23}^2 +$$

$$12 \mathcal{D} F_{21}^2 F_{23}^2 + 4 \mathcal{D} F_{21}^4 F_{23}^2 + 16 \mathcal{D} (-1 + F_{22}) F_{23}^2 + 8 \mathcal{D} F_{21}^2 (-1 + F_{22}) F_{23}^2 + 8 \mathcal{D} (-1 + F_{22})^2 F_{23}^2 +$$

$$4 \mathcal{D} F_{21}^2 (-1 + F_{22})^2 F_{23}^2 + 4 \mathcal{D} F_{23}^4 + 2 \mathcal{D} F_{21}^2 F_{23}^4 + 2 \mathcal{A} F_{21} F_{23} F_{31} + 16 \mathcal{D} F_{21} F_{23} F_{31} +$$

$$8 \mathcal{D} F_{21}^3 F_{23} F_{31} + 5 \mathcal{A} F_{31}^2 + 8 \mathcal{D} F_{31}^2 + 4 \mu F_{31}^2 + 2 \mathcal{A} F_{21}^2 F_{31}^2 + 32 \mathcal{D} F_{21}^2 F_{31}^2 + 6 \mathcal{D} F_{21}^4 F_{31}^2 +$$

$$8 \mathcal{D} F_{21}^2 (-1 + F_{22}) F_{31}^2 + 8 \mathcal{D} (-1 + F_{22})^2 F_{31}^2 + 4 \mathcal{D} F_{21}^2 (-1 + F_{22})^2 F_{31}^2 + 8 \mathcal{D} (-1 + F_{22})^3 F_{31}^2 +$$

$$2 \mathcal{D} (-1 + F_{22})^4 F_{31}^2 + 4 \mathcal{D} F_{23}^2 F_{31}^2 + 4 \mathcal{D} F_{21}^2 F_{23}^2 F_{31}^2 + 8 \mathcal{D} (-1 + F_{22}) F_{23}^2 F_{31}^2 +$$

$$4 \mathcal{D} (-1 + F_{22})^2 F_{23}^2 F_{31}^2 + 2 \mathcal{D} F_{23}^4 F_{31}^2 + 8 \mathcal{D} F_{21} F_{23} F_{31}^3 + \mathcal{A} F_{31}^4 + 16 \mathcal{D} F_{31}^4 +$$

$$6 \mathcal{D} F_{21}^2 F_{31}^4 + 2 \mathcal{D} F_{31}^6 + 16 \mathcal{D} F_{23} F_{32} + 8 \mathcal{D} F_{21}^2 F_{23} F_{32} + 16 \mathcal{D} (-1 + F_{22}) F_{23} F_{32} +$$

$$8 \mathcal{D} F_{21}^2 (-1 + F_{22}) F_{23} F_{32} + 2 \mathcal{A} F_{21} F_{31} F_{32} + 16 \mathcal{D} F_{21} F_{31} F_{32} + 8 \mathcal{D} F_{21}^3 F_{31} F_{32} +$$

$$2 \mathcal{A} F_{21} (-1 + F_{22}) F_{31} F_{32} + 16 \mathcal{D} F_{21} (-1 + F_{22}) F_{31} F_{32} + 8 \mathcal{D} F_{21}^3 (-1 + F_{22}) F_{31} F_{32} +$$

$$8 \mathcal{D} F_{23} F_{31}^2 F_{32} + 8 \mathcal{D} (-1 + F_{22}) F_{23} F_{31}^2 F_{32} + 8 \mathcal{D} F_{21} F_{31}^3 F_{32} + 8 \mathcal{D} F_{21} (-1 + F_{22}) F_{31}^3 F_{32} +$$

$$8 \mathcal{D} F_{32}^2 + 4 \mathcal{D} F_{21}^2 F_{32}^2 + 16 \mathcal{D} (-1 + F_{22}) F_{32}^2 + 8 \mathcal{D} F_{21}^2 (-1 + F_{22}) F_{32}^2 + 8 \mathcal{D} (-1 + F_{22})^2 F_{32}^2 +$$

$$4 \mathcal{D} F_{21}^2 (-1 + F_{22})^2 F_{32}^2 + \mathcal{A} F_{31}^2 F_{32}^2 + 12 \mathcal{D} F_{31}^2 F_{32}^2 + 4 \mathcal{D} F_{21}^2 F_{31}^2 F_{32}^2 + 8 \mathcal{D} (-1 + F_{22}) F_{31}^2 F_{32}^2 +$$

$$4 \mathcal{D} (-1 + F_{22})^2 F_{31}^2 F_{32}^2 + 4 \mathcal{D} F_{31}^4 F_{32}^2 + 4 \mathcal{D} F_{32}^4 + 2 \mathcal{D} F_{21}^2 F_{32}^4 + 2 \mathcal{D} F_{31}^2 F_{32}^4 +$$

$$F_{12}^4 \left(\mathcal{A} + 16 \mathcal{D} + 6 \mathcal{D} F_{13}^2 + 2 \mathcal{D} F_{21}^2 + 8 \mathcal{D} (-1 + F_{22}) + 4 \mathcal{D} (-1 + F_{22})^2 + 2 \mathcal{D} F_{31}^2 + 4 \mathcal{D} F_{32}^2\right) +$$

$$F_{13}^4 \left(\mathcal{A} + 16 \mathcal{D} + 2 \mathcal{D} F_{21}^2 + 4 \mathcal{D} F_{23}^2 + 2 \mathcal{D} F_{31}^2 + 8 \mathcal{D} (-1 + F_{33}) + 4 \mathcal{D} (-1 + F_{33})^2\right) +$$

$$16 \mathcal{D} F_{23}^2 (-1 + F_{33}) + 8 \mathcal{D} F_{21}^2 F_{23}^2 (-1 + F_{33}) + 2 \mathcal{A} F_{21} F_{23} F_{31} (-1 + F_{33}) +$$

$$16 \mathcal{D} F_{21} F_{23} F_{31} (-1 + F_{33}) + 8 \mathcal{D} F_{21}^3 F_{23} F_{31} (-1 + F_{33}) + 2 \mathcal{A} F_{31}^2 (-1 + F_{33}) +$$

$$16 \mathcal{D} F_{31}^2 (-1 + F_{33}) + 8 \mathcal{D} F_{21}^2 F_{31}^2 (-1 + F_{33}) + 8 \mathcal{D} F_{23}^2 F_{31}^2 (-1 + F_{33}) + 8 \mathcal{D} F_{21} F_{23} F_{31}^3$$

$$(-1 + F_{33}) + 8 \mathcal{D} F_{31}^4 (-1 + F_{33}) + 16 \mathcal{D} F_{23} F_{32} (-1 + F_{33}) + 8 \mathcal{D} F_{21}^2 F_{23} F_{32} (-1 + F_{33}) +$$

$$16 \mathcal{D} (-1 + F_{22}) F_{23} F_{32} (-1 + F_{33}) + 8 \mathcal{D} F_{21}^2 (-1 + F_{22}) F_{23} F_{32} (-1 + F_{33}) +$$

$$8 \mathcal{D} F_{23} F_{31}^2 F_{32} (-1 + F_{33}) + 8 \mathcal{D} (-1 + F_{22}) F_{23} F_{31}^2 F_{32} (-1 + F_{33}) + 16 \mathcal{D} F_{32}^2 (-1 + F_{33}) +$$

$$8 \mathcal{D} F_{21}^2 F_{32}^2 (-1 + F_{33}) + 8 \mathcal{D} F_{31}^2 F_{32}^2 (-1 + F_{33}) + 16 \mathcal{D} (-1 + F_{33})^2 + 8 \mathcal{D} F_{21}^2 (-1 + F_{33})^2 +$$

$$8 \mathcal{D} F_{23}^2 (-1 + F_{33})^2 + 4 \mathcal{D} F_{21}^2 F_{23}^2 (-1 + F_{33})^2 + \mathcal{A} F_{31}^2 (-1 + F_{33})^2 + 16 \mathcal{D} F_{31}^2 (-1 + F_{33})^2 +$$

$$4 \mathcal{D} F_{21}^2 F_{31}^2 (-1 + F_{33})^2 + 4 \mathcal{D} F_{23}^2 F_{31}^2 (-1 + F_{33})^2 + 4 \mathcal{D} F_{31}^4 (-1 + F_{33})^2 + 8 \mathcal{D} F_{32}^2 (-1 + F_{33})^2 +$$

$$4 \mathcal{D} F_{21}^2 F_{32}^2 (-1 + F_{33})^2 + 4 \mathcal{D} F_{31}^2 F_{32}^2 (-1 + F_{33})^2 + 16 \mathcal{D} (-1 + F_{33})^3 + 8 \mathcal{D} F_{21}^2 (-1 + F_{33})^3 +$$

$$8 \mathcal{D} F_{31}^2 (-1 + F_{33})^3 + 4 \mathcal{D} (-1 + F_{33})^4 + 2 \mathcal{D} F_{21}^2 (-1 + F_{33})^4 + 2 \mathcal{D} F_{31}^2 (-1 + F_{33})^4 +$$

$$24 \mathcal{D} F_{13}^3 (F_{21} F_{23} + F_{31} F_{33}) + 2 F_{13} \left(3 \mathcal{A} + 8 \mathcal{D} + 12 \mathcal{D} F_{21}^2 + 12 \mathcal{D} F_{31}^2\right) (F_{21} F_{23} + F_{31} F_{33}) +$$

$$F_{12}^2 \left(5 \mathcal{A} + 8 \mathcal{D} + 4 \mu + 6 \mathcal{D} F_{13}^4 + 2 \mathcal{D} F_{21}^4 + 2 \mathcal{A} (-1 + F_{22}) + 16 \mathcal{D} (-1 + F_{22}) + \mathcal{A} (-1 + F_{22})^2 +\right.$$

$$
\begin{aligned}
&16\,\mathcal{D}\,(-1+F_{22})^2+8\,\mathcal{D}\,(-1+F_{22})^3+2\,\mathcal{D}\,(-1+F_{22})^4+4\,\mathcal{D}\,F_{23}^2+8\,\mathcal{D}\,(-1+F_{22})\,F_{23}^2+\\
&4\,\mathcal{D}\,(-1+F_{22})^2\,F_{23}^2+2\,\mathcal{D}\,F_{23}^4+\mathcal{A}\,F_{31}^2+24\,\mathcal{D}\,F_{31}^2+8\,\mathcal{D}\,(-1+F_{22})\,F_{31}^2+\\
&4\,\mathcal{D}\,(-1+F_{22})^2\,F_{31}^2+2\,\mathcal{D}\,F_{31}^4+8\,\mathcal{D}\,F_{23}\,F_{32}+8\,\mathcal{D}\,(-1+F_{22})\,F_{23}\,F_{32}+\mathcal{A}\,F_{32}^2+\\
&12\,\mathcal{D}\,F_{32}^2+8\,\mathcal{D}\,(-1+F_{22})\,F_{32}^2+4\,\mathcal{D}\,(-1+F_{22})^2\,F_{32}^2+16\,\mathcal{D}\,F_{31}^2\,F_{32}^2+2\,\mathcal{D}\,F_{32}^4+\\
&F_{21}^2\left(\mathcal{A}+32\,\mathcal{D}+32\,\mathcal{D}\,(-1+F_{22})+16\,\mathcal{D}\,(-1+F_{22})^2+4\,\mathcal{D}\,F_{23}^2+4\,\mathcal{D}\,F_{31}^2+4\,\mathcal{D}\,F_{32}^2\right)+\\
&2\,F_{13}^2\left(\mathcal{A}+16\,\mathcal{D}+2\,\mathcal{D}\,F_{21}^2+4\,\mathcal{D}\,(-1+F_{22})+2\,\mathcal{D}\,(-1+F_{22})^2+2\,\mathcal{D}\,F_{23}^2+2\,\mathcal{D}\,F_{31}^2+2\,\mathcal{D}\,F_{32}^2+\right.\\
&\quad\left.4\,\mathcal{D}\,(-1+F_{33})+2\,\mathcal{D}\,(-1+F_{33})^2\right)+8\,\mathcal{D}\,F_{23}^2\,(-1+F_{33})+8\,\mathcal{D}\,F_{31}^2\,(-1+F_{33})+8\,\mathcal{D}\,F_{23}\,F_{32}\\
&(-1+F_{33})+8\,\mathcal{D}\,(-1+F_{22})\,F_{23}\,F_{32}\,(-1+F_{33})+8\,\mathcal{D}\,F_{32}^2\,(-1+F_{33})+8\,\mathcal{D}\,(-1+F_{33})^2+\\
&4\,\mathcal{D}\,F_{23}^2\,(-1+F_{33})^2+4\,\mathcal{D}\,F_{31}^2\,(-1+F_{33})^2+4\,\mathcal{D}\,F_{32}^2\,(-1+F_{33})^2+8\,\mathcal{D}\,(-1+F_{33})^3+\\
&2\,\mathcal{D}\,(-1+F_{33})^4+8\,\mathcal{D}\,F_{21}\,F_{31}\,(3\,F_{22}\,F_{32}+F_{23}\,F_{33})+24\,\mathcal{D}\,F_{13}\,(F_{21}\,F_{23}+F_{31}\,F_{33})\big)+\\
&8\,\mathcal{D}\,F_{12}^3\,(3\,F_{21}\,F_{22}+3\,F_{31}\,F_{32}+F_{13}\,(F_{22}\,F_{23}+F_{32}\,F_{33}))+\\
&F_{13}^2\left(5\,\mathcal{A}+8\,\mathcal{D}+4\,\mu+2\,\mathcal{D}\,F_{21}^4+8\,\mathcal{D}\,(-1+F_{22})^3+2\,\mathcal{D}\,(-1+F_{22})^4+\mathcal{A}\,F_{23}^2+12\,\mathcal{D}\,F_{23}^2+\right.\\
&2\,\mathcal{D}\,F_{23}^4+\mathcal{A}\,F_{31}^2+32\,\mathcal{D}\,F_{31}^2+4\,\mathcal{D}\,F_{23}^2\,F_{31}^2+2\,\mathcal{D}\,F_{31}^4+8\,\mathcal{D}\,F_{23}\,F_{32}+4\,\mathcal{D}\,F_{32}^2+\\
&4\,\mathcal{D}\,F_{31}^2\,F_{32}^2+2\,\mathcal{D}\,F_{32}^4+4\,\mathcal{D}\,(-1+F_{22})^2\left(2+F_{23}^2+F_{32}^2\right)+F_{21}^2\left(\mathcal{A}+24\,\mathcal{D}+8\,\mathcal{D}\,(-1+F_{22})+\right.\\
&\quad\left.4\,\mathcal{D}\,(-1+F_{22})^2+16\,\mathcal{D}\,F_{23}^2+4\,\mathcal{D}\,F_{31}^2+8\,\mathcal{D}\,(-1+F_{33})+4\,\mathcal{D}\,(-1+F_{33})^2\right)+\\
&2\,\mathcal{A}\,(-1+F_{33})+16\,\mathcal{D}\,(-1+F_{33})+8\,\mathcal{D}\,F_{23}^2\,(-1+F_{33})+32\,\mathcal{D}\,F_{31}^2\,(-1+F_{33})+\\
&8\,\mathcal{D}\,F_{23}\,F_{32}\,(-1+F_{33})+8\,\mathcal{D}\,F_{32}^2\,(-1+F_{33})+\mathcal{A}\,(-1+F_{33})^2+16\,\mathcal{D}\,(-1+F_{33})^2+\\
&4\,\mathcal{D}\,F_{23}^2\,(-1+F_{33})^2+16\,\mathcal{D}\,F_{31}^2\,(-1+F_{33})^2+4\,\mathcal{D}\,F_{32}^2\,(-1+F_{33})^2+8\,\mathcal{D}\,(-1+F_{33})^3+\\
&\left.2\,\mathcal{D}\,(-1+F_{33})^4+8\,\mathcal{D}\,F_{21}\,F_{31}\,(F_{22}\,F_{32}+3\,F_{23}\,F_{33})+8\,\mathcal{D}\,(-1+F_{22})\left(F_{23}^2+F_{32}^2+F_{23}\,F_{32}\,F_{33}\right)\right)+\\
&2\,F_{12}\left(12\,\mathcal{D}\,F_{21}^3\,F_{22}+12\,\mathcal{D}\,F_{13}^2\,F_{31}\,F_{32}+F_{31}\left(3\,\mathcal{A}+8\,\mathcal{D}+12\,\mathcal{D}\,F_{31}^2\right)F_{32}+4\,\mathcal{D}\,F_{13}^3\right.\\
&\quad(F_{22}\,F_{23}+F_{32}\,F_{33})+F_{13}\left(F_{22}\,F_{23}\left(\mathcal{A}+8\,\mathcal{D}+4\,\mathcal{D}\,F_{31}^2\right)+\left(\mathcal{A}+8\,\mathcal{D}+12\,\mathcal{D}\,F_{31}^2\right)F_{32}\,F_{33}\right)+F_{21}\\
&\quad\left(12\,\mathcal{D}\,F_{13}^2\,F_{22}+F_{22}\left(3\,\mathcal{A}+8\,\mathcal{D}+12\,\mathcal{D}\,F_{31}^2\right)+8\,\mathcal{D}\,F_{13}\,F_{31}\,(F_{23}\,F_{32}+F_{33}+(-1+F_{22})\,F_{33})\right)+\\
&\left.4\,\mathcal{D}\,F_{21}^2\,(3\,F_{31}\,F_{32}+F_{13}\,(3\,F_{22}\,F_{23}+F_{32}\,F_{33}))\right)\big)\big)
\end{aligned}
$$

In[ ]:=

```
TableForm[f_SS[[ ;; , 4 ;;]], TableDirections → Row]
```

Out[ ]//TableForm=

v_1	0	0
0	v_1	0
0	0	v_1
v_2	0	0
0	v_2	0
0	0	v_2
v_3	0	0
0	v_3	0
0	0	v_3

Flux Jacobians

In[ ]:=
```
ASS = Map[Transpose[gradq[#]] &, fSS];
```

In[ ]:=
```
Dimensions[ASS]
```

Out[ ]=
```
{3, 12, 12}
```

In[ ]:=
```
TableForm[{ ASS[[1, 4 ;;, ;;]], ALE[[1, 4 ;;, ;;]]} , TableDirections → Row]
```

Out[ ]//TableForm=
```
1 0 0 0 0 0 0 0 0 0 0 0  1 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0  0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0  0 0 0 0 0 0 0 0 0 0 0 0
0 1 0 0 0 0 0 0 0 0 0 0  0 1 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0  0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0  0 0 0 0 0 0 0 0 0 0 0 0
0 0 1 0 0 0 0 0 0 0 0 0  0 0 1 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0  0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0  0 0 0 0 0 0 0 0 0 0 0 0
```

In[ ]:=
```
TableForm[{ ASS[[2, 4 ;;, ;;]], ALE[[2, 4 ;;, ;;]]} , TableDirections → Row]
```

Out[ ]//TableForm=
```
0 0 0 0 0 0 0 0 0 0 0 0  0 0 0 0 0 0 0 0 0 0 0 0
1 0 0 0 0 0 0 0 0 0 0 0  1 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0  0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0  0 0 0 0 0 0 0 0 0 0 0 0
0 1 0 0 0 0 0 0 0 0 0 0  0 1 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0  0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0  0 0 0 0 0 0 0 0 0 0 0 0
0 0 1 0 0 0 0 0 0 0 0 0  0 0 1 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0  0 0 0 0 0 0 0 0 0 0 0 0
```

In[ ]:=
```
TableForm[{ A_SS[[3, 4 ;;,  ;;]], A_LE[[3, 4 ;;,  ;;]]} , TableDirections → Row]
```

Out[ ]//TableForm=

0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0
1 0 0 0 0 0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0
0 1 0 0 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0
0 0 1 0 0 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0 0 0 0

Eigensystem - eigenvalues

In[ ]:=
```
ΛR = Map[Eigensystem[#] &, A];
Λ = Table[ΛR[[i, 1]], {i, d}];
R = Table[ΛR[[i, 2]], {i, d}];
TableForm[Λ]
```

Out[ ]//TableForm=

0	0	0	0	0	0	$-c_S$	$-c_S$	c_S	c_S	$-c_P$	c_P
0	0	0	0	0	0	$-c_S$	$-c_S$	c_S	c_S	$-c_P$	c_P
0	0	0	0	0	0	$-c_S$	$-c_S$	c_S	c_S	$-c_P$	c_P

Eigensystem - eigenvectors (displayed as rows in Mathematica)

In[ ]:=
```
TableForm[R[[1]]]
```

Out[ ]//TableForm=

0	0	0	$-\frac{\lambda}{\lambda+2\mu}$	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	-1	0	0	0	1	0	0
0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	$-\frac{\lambda}{\lambda+2\mu}$	0	0	0	1	0	0	0	0
0	0	0	0	-1	0	1	0	0	0	0	0
0	0	$-c_S$	0	0	0	0	0	0	1	0	0
0	$-c_S$	0	0	0	0	1	0	0	0	0	0
0	0	c_S	0	0	0	0	0	0	1	0	0
0	c_S	0	0	0	0	1	0	0	0	0	0
$-c_P$	0	0	1	0	0	0	0	0	0	0	0
c_P	0	0	1	0	0	0	0	0	0	0	0

In[ ]:=
```
TableForm[R[[2]]]
```
Out[ ]//TableForm=

0	0	0	-1	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	-1	0	1	0
0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	$-\frac{\lambda+2\mu}{\lambda}$	0	0	0	1	0	0	0	0
0	0	0	0	-1	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0
0	0	$-c_S$	0	0	0	0	0	0	0	1	0
$-c_S$	0	0	0	1	0	0	0	0	0	0	0
0	0	c_S	0	0	0	0	0	0	0	1	0
c_S	0	0	0	1	0	0	0	0	0	0	0
0	$-c_P$	0	0	0	0	0	1	0	0	0	0
0	c_P	0	0	0	0	0	1	0	0	0	0

In[ ]:=
```
TableForm[R[[3]]]
```
Out[ ]//TableForm=

0	0	0	$-\frac{\lambda+2\mu}{\lambda}$	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	-1	0	1	0
0	0	0	0	0	-1	0	0	0	1	0	0
0	0	0	-1	0	0	0	1	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0
0	$-c_S$	0	0	0	0	0	0	1	0	0	0
$-c_S$	0	0	0	0	1	0	0	0	0	0	0
0	c_S	0	0	0	0	0	0	1	0	0	0
c_S	0	0	0	0	1	0	0	0	0	0	0
0	0	$-c_P$	0	0	0	0	0	0	0	0	1
0	0	c_P	0	0	0	0	0	0	0	0	1

Verify eigensystem

In[ ]:=
```
TableForm[Table[A[[i]].R[[i, α]] - Λ[[i, α]] × R[[i, α]] /. {cP → Sqrt[λ + 2 μ], cS → Sqrt[μ]},
  {α, nq}, {i, d}], TableDirections → Row]
```
Out[ ]//TableForm=

0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0 0

Decomposition of δq onto eigenbasis, R c = δq

```
In[ ]:= δq = Table[δq_α, {α, nq}]; TableForm[δq, TableDirections → Row]
Out[ ]//TableForm=
δq_1  δq_2  δq_3  δq_4  δq_5  δq_6  δq_7  δq_8  δq_9  δq_10  δq_11  δq_12
```

Define coefficients c

```
In[ ]:= Table[c_α, {α, nq}]; TableForm[c, TableDirections → Row]
Out[ ]//TableForm=
c
```

```
In[ ]:= c = Table[LinearSolve[Transpose[R[[i]]], δq], {i, d}];
```

Select propagating waves (place arbitrary positive values for wave velocities into eigenvalues)

```
In[ ]:= Λ12 = Λ /. {c_S → 1, c_P → 2};
iW = Table[Select[Table[α, {α, nq}], Λ12[[i, #]] ≠ 0 &], {i, d}]; TableForm[iW]
Out[ ]//TableForm=
7    8    9    10    11    12
7    8    9    10    11    12
7    8    9    10    11    12
```

Number of propagating waves. Should be the same for all d spatial dimensions.

```
In[ ]:= nW = Length[iW[[1]]]
Out[ ]= 6
```

```
In[ ]:= λW = Table[Λ[[i, iW[[i]]]], {i, d}]; TableForm[λW]
Out[ ]//TableForm=
-c_S    -c_S    c_S    c_S    -c_P    c_P
-c_S    -c_S    c_S    c_S    -c_P    c_P
-c_S    -c_S    c_S    c_S    -c_P    c_P
```

```
In[ ]:= W = Table[R[[i, iW[[i]]]], {i, d}]; TableForm[W[[1]]]
Out[ ]//TableForm=
0      0      -c_S   0   0   0   0   0   0   1   0   0
0      -c_S   0      0   0   0   1   0   0   0   0   0
0      0      c_S    0   0   0   0   0   0   1   0   0
0      c_S    0      0   0   0   1   0   0   0   0   0
-c_P   0      0      1   0   0   0   0   0   0   0   0
c_P    0      0      1   0   0   0   0   0   0   0   0
```

In[ ]:=
```
TableForm[W[[2]]]
```
Out[ ]//TableForm=

0	0	$-c_S$	0	0	0	0	0	0	0	1	0
$-c_S$	0	0	0	1	0	0	0	0	0	0	0
0	0	c_S	0	0	0	0	0	0	0	1	0
c_S	0	0	0	1	0	0	0	0	0	0	0
0	$-c_P$	0	0	0	0	0	1	0	0	0	0
0	c_P	0	0	0	0	0	1	0	0	0	0

In[ ]:=
```
TableForm[W[[3]]]
```
Out[ ]//TableForm=

0	$-c_S$	0	0	0	0	0	0	1	0	0	0
$-c_S$	0	0	0	0	1	0	0	0	0	0	0
0	c_S	0	0	0	0	0	0	1	0	0	0
c_S	0	0	0	0	1	0	0	0	0	0	0
0	0	$-c_P$	0	0	0	0	0	0	0	0	1
0	0	c_P	0	0	0	0	0	0	0	0	1

In[ ]:=
```
cW = Table[c[[i, iW[[i]]]], {i, d}]; TableForm[cW]
```
Out[ ]//TableForm=

$\frac{-\delta q_3+c_S\,\delta q_6+c_S\,\delta q_{10}}{2\,c_S}$	$\frac{-\delta q_2+c_S\,\delta q_5+c_S\,\delta q_7}{2\,c_S}$	$\frac{\delta q_3+c_S\,\delta q_6+c_S\,\delta q_{10}}{2\,c_S}$	$\frac{\delta q_2+c_S\,\delta q_5+c_S\,\delta q_7}{2\,c_S}$	$\frac{-\lambda\,\delta q_1-2\,\mu\,\delta q_1+\lambda\,c_P\,\delta q_4+2}{2\,(\lambda+}$
$\frac{-\delta q_3+c_S\,\delta q_9+c_S\,\delta q_{11}}{2\,c_S}$	$\frac{-\delta q_1+c_S\,\delta q_5+c_S\,\delta q_7}{2\,c_S}$	$\frac{\delta q_3+c_S\,\delta q_9+c_S\,\delta q_{11}}{2\,c_S}$	$\frac{\delta q_1+c_S\,\delta q_5+c_S\,\delta q_7}{2\,c_S}$	$\frac{-\lambda\,\delta q_2-2\,\mu\,\delta q_2+\lambda\,c_P\,\delta q_4+\lambda}{2\,(\lambda+}$
$\frac{-\delta q_2+c_S\,\delta q_9+c_S\,\delta q_{11}}{2\,c_S}$	$\frac{-\delta q_1+c_S\,\delta q_6+c_S\,\delta q_{10}}{2\,c_S}$	$\frac{\delta q_2+c_S\,\delta q_9+c_S\,\delta q_{11}}{2\,c_S}$	$\frac{\delta q_1+c_S\,\delta q_6+c_S\,\delta q_{10}}{2\,c_S}$	$\frac{-\lambda\,\delta q_3-2\,\mu\,\delta q_3+\lambda\,c_P\,\delta q_4+\lambda}{2\,(\lambda+}$

In[ ]:=
```
Directory[]
```
Out[ ]=
```
C:\Users\sorin\Documents
```

In[ ]:=
```
Save["linelasWaves", {q, nW, W, cW}];
```