SciComp Practice Exam 5/26/14

*Answer the following questions explaining all steps that lead to a solution. Results presented without motivation will **not** receive any credit.*

1. Transform the integral equation

$$u(x) - \mu \int_0^1 K(x,y)\, f(y)\, \mathrm{d}y = g(x)$$

into a linear system using the Simpson quadrature formula. Give an estimate of the accuracy of the solution and the computational workload with increasing number of discretization points.

2. The discrete Fourier transform (DFT) of a vector $x \in \mathbb{R}^N$ is $X = \mathcal{F}x \in \mathbb{C}^m$, with components

$$X_k = \sum_{j=0}^{N-1} \exp\left(-\frac{2\pi i j k}{N}\right) x_j, k = 0, \ldots, N-1, i = \sqrt{-1}.$$

Devise an algorithm to approximate the solution of $x'' + a x' + b x = f$ by use of the DFT. Estimate the computational workload.

3. What is the limit of the sequence

$$x_{n+1} = \frac{x_n(x_n^2 + 3R)}{3x_n^2 + R}? \tag{1}$$

State the equation for which (1) is a root-finding algorithm. At what rate of convergence does the sequence approach its limit?

4. Prove or disprove: For $A \in \mathbb{R}^{m \times m}$, $A = A^T$, A nonnegative definite iff all leading principal minors of A have non-negative determinant.

5. Consider the bound

$$\frac{\|\delta x\|}{\|x\|} \leqslant \kappa(A) \frac{\|\delta b\|}{\|b\|} \tag{2}$$

on effect of rhs term perturbations δb upon solutions of the linear system $Ax = b$ (A nonsingular, $b \neq 0$). For what $\delta b \neq 0$ does (2) become an equality.

6. Analyze the performance of

$$x_n - x_{n-1} = h f\left(t_{n-1} + \frac{1}{2}h, \frac{1}{2}(x_n + x_{n-1})\right)$$

as a method of finding solutions to stiff ODEs (Hint: consider the model ODE $x' = \lambda x$, $\lambda < 0$)