Fall 2013 Scientific Computation Comprehensive Examination

Answer 5 questions of your choice explaining all steps that lead to a solution. Partial credit will be awarded for presenting a viable solution strategy. No credit will be given to computations presented without motivation.

1. Use the SVD to find matrix $B \in \mathbb{R}^{m \times m}$ that has the same determinant absolute value as $A \in \mathbb{R}^{m \times m}$, and best approximates A in the Frobenius norm,

$$\min_{\det B = |\det A|} \|A - B\|_F .$$

2. The Lanczos algorithm is a simplification of the Arnoldi iteration (Algorithm 1 below) for general $A \in \mathbb{C}^{m \times m}$, $b \in \mathbb{C}^m$

Algorithm 1

b arbitrary, $q_1 = b/\|b\|$
for $n = 1, 2, 3, \ldots$
  $v = A q_n$
  for $j = 1$ to n
    $h_{jn} = q_j^* v$
    $v = v - h_{jn} q_j$
  $h_{n+1,n} = \|v\|$
  $q_{n+1} = v / h_{n+1,n}$

for the special case when $A \in \mathbb{R}^{m \times m}$ and $A = A^T$ (symmetric A).

a) Write out the Lanczos algorithm in a computationally efficient form.

b) Present an analogous algorithm for the case $A = -A^T$ (skew-symmetric A).

3. Let f be the nonlinear function with zero x^* given by

$$f(x, y) = \begin{pmatrix} \cos x + e^y - 2 \\ e^{-x} + \sin y - 1 \end{pmatrix}, \quad x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Consider the iteration

$$A x_{n+1} = -f(x_n), \quad A = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}.$$

a. Assuming x_0 is close to x^*, determine for which α the iterations converge.

b. Write out the Newton's method for this function and comment on its convergence.

4. Consider the following scheme for solving the ODE $y'(t) = f(y(t))$,

$$y_{n+1} = y_n + h\,(\alpha\, f(y_n) + \beta\, f(y_{n-1})).$$

Assume that the mesh is uniform with step size h, and $y_0 = y(0)$, $y_1 = y(h)$. This scheme is derived from the numerical integration formula

$$y_{n+1} = y_n + \int_{t_n}^{t^{n+1}} p(t)\,dt,$$

where $p(t)$ is the interpolating polynomial satisfying $p(t_n) = f_n = f(t_n,\ y_n)$ and $p(t_{n-1}) = f_{n-1} = f(t_{n-1}, y_{n-1})$.

   a. Find α and β.

   b. What is the local truncation error of this method?

   c. Is this method convergent?

5. Consider the problem of constructing a polynomial approximation g of an analytic function $f\colon [a, b] \to \mathbb{R}$, using 4 precomputed values $(x_i, y_i = f(x_i))$, $i = 1, \ldots, 4$.

   a. How should x_i be chosen to ensure that the error

$$\varepsilon = \max_{a \leqslant x \leqslant b} |f(x) - g(x)|$$

   is minimized?

   b. Present a tight upper bound for the error ε, and evaluate it for $f(x) = \exp(x)$ on $[a, b] = [-1, 1]$. Is the upper bound attained in this case?

6. Consider the numerical quadrature of integrals of form

$$\int_0^h \sqrt{x}\, f(x)\,\mathrm{d}x = A f(ah) + B f(bh) + C h^p, \tag{1}$$

   with $h > 0$, and $f\colon [0, h] \to \mathbb{R}$ an analytic function.

   a. Determine a, b, A, B corresponding to two-point Gauss-Legendre quadrature

$$\int_{-1}^{+1} F(x)\,\mathrm{d}x \cong F\left(-\frac{1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right).$$

   Present possible deficiencies of this approach (perhaps based upon a counter-example).

   b. Determine alternative values a, b, A, B such that the order of the quadrature method p is as high as possible. Compute C for this case.