Scientific Computation Comprehensive Examination Practice Questions

Answer the following questions explaining all steps that lead to a solution. Partial credit will be awarded for presenting a viable solution strategy. No credit will be given to computations presented without motivation. Your goal is to present skill in formulating precise mathematical statements, and demonstrate understanding of theoretical material.

1. Find the solution of the difference equation

$$x_{n+4} - \frac{3}{2}x_{n+3} + \frac{3}{4}x_{n+1} - \frac{1}{4}x_n = 0,$$

   with initial conditions $x_0 = 0, x_1 = 0, x_2 = 1, x_3 = 1$. Comment on the the effect upon x_n of a small perturbation δ on the initial condition $x_3' = 1 + \delta$.

2. Determine the quadrature nodes and weights of the Gaussian approximation formula

$$\int_0^1 x^4 \, f(x) \, \mathrm{d}x \cong A_0 f(x_0) + A_1 f(x_1).$$

3. Solve the equation $x^3 + 2x^2 + x - 2 = 0$, finding the roots to 3 significant digits.

4. Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, with $m > n$ have QR factorization $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$, $\boldsymbol{Q} \in \mathbb{R}^{m \times m}$, $\boldsymbol{R} \in \mathbb{R}^{m \times n}$. Let $\boldsymbol{P} \in \mathbb{R}^{m \times n}$ be the matrix formed by taking the first n columns of $\boldsymbol{Q}$, and let $\boldsymbol{S} \in \mathbb{R}^{n \times n}$ be the matrix formed from the first n rows of $\boldsymbol{R}$. Prove that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{S}$.

5. Find the pseudoinverse of $\boldsymbol{u}\boldsymbol{v}^*$, $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^m$.