Scientific Computation Comprehensive Examination Practice Questions

Answer the following questions explaining all steps that lead to a solution. Partial credit will be awarded for presenting a viable solution strategy. No credit will be given to computations presented without motivation. Your goal is to present skill in formulating precise mathematical statements, and demonstrate understanding of theoretical material.

1. Let $S_1(x)$ denote the cubic spline interpolation of data $\mathcal{D}=\{(t_i,\ f_i),\, i=0,1,\dots,n\}$, $t_0=a$, $t_n=b$, that samples $f\in C^2[a,b]$, with end conditions $S'(a)=f'(a)$, $S'(b)=f'(b)$. Let $S_2(x)$ denote the cubic spline interpolation of the same data with end conditions $S''(a)=0$, $S''(b)=0$.

   a) Compute $\|S_1''-S_2''\|_2$.

   b) Is S_1 equal to S_2?

2. Consider the linear system $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$, with $\boldsymbol{A}=\boldsymbol{A}^T$, and let $q(\boldsymbol{x})=\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}-2\boldsymbol{x}^T\boldsymbol{b}$.

   a) State the steepest descent algorithm to solve the linear system.

   b) Let $\boldsymbol{x}_k$ denote the k^{th} steepest descent approximant of the solution, and $\boldsymbol{r}_k=\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x}_k$ the associated residual. Prove

   $$q(\boldsymbol{x}_{k+1})=q(\boldsymbol{x}_k)-\frac{\|\boldsymbol{r}_k\|^4}{\boldsymbol{r}_k^T\boldsymbol{A}\boldsymbol{r}_k}.$$

   What are the implications of the above for steepest descent convergence?

3. Determine the minimum number of subintervals needed to approximate

   $$I=\int_1^2\left(x+e^{-x^2}\right)\mathrm{d}x$$

   to an accuracy of at least 5×10^{-8} using the trapezoid rule.