Scientific Computation Comprehensive Examination Practice Questions

Answer the following questions explaining all steps that lead to a solution. Partial credit will be awarded for presenting a viable solution strategy. No credit will be given to computations presented without motivation. Your goal is to present skill in formulating precise mathematical statements, and demonstrate understanding of theoretical material.

1. Prove that no Gaussian quadrature formula with n nodes can be exact for all polynomials of degree $2n$.

2. Let $x_j = 2\pi j / N$, for $j = 0, 1, \ldots, N-1$, and define the scalar product

$$\langle f, g \rangle_N = \frac{1}{N} \sum_{j=0}^{N-1} f_j \bar{g}_j$$

with $f_j = f(x_j)$, $g_j = g(x_j)$, $\bar{z}$ the complex conjugate of $z \in \mathbb{C}$. Define column vectors $\boldsymbol{e}_k, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^N$, $k = 0, 1, \ldots, N-1$, with components

$$e_{jk} = \exp(i k x_j), u_j = f(x_j), v_j = \langle f, \boldsymbol{e}_j \rangle_N, j = 0, 1, \ldots, N-1,$$

and the matrix $\boldsymbol{E} = \frac{1}{N}(\ \boldsymbol{e}_0 \ \ \boldsymbol{e}_1 \ \ \ldots \ \ \boldsymbol{e}_{N-1} \ ) \in \mathbb{C}^{N \times N}$.

   a) Prove that $\boldsymbol{v} = \boldsymbol{E}\boldsymbol{u}$.

   b) How many distinct elements does $\boldsymbol{E}$ have? Is $\boldsymbol{E}$ symmetric? Is $\boldsymbol{E}$ unitary?

   c) Show $\|\boldsymbol{v}\| = \frac{1}{\sqrt{N}} \|\boldsymbol{u}\|$, in the Euclidean norm on $\mathbb{C}^N$.

   d) Compute the singular value decomposition of $\boldsymbol{E}$.

3. Present an analysis of the multistep method defined by

$$x_n - 3x_{n-1} + 2x_{n-2} = h(f_n + 2f_{n-1} + f_{n-2} - 2f_{n-3}),$$

for solving the ODE $x'(t) = f(t, x)$, with $t_n = nh$, $x_n = x(t_n)$, $f_n = f(t_n, x_n)$.